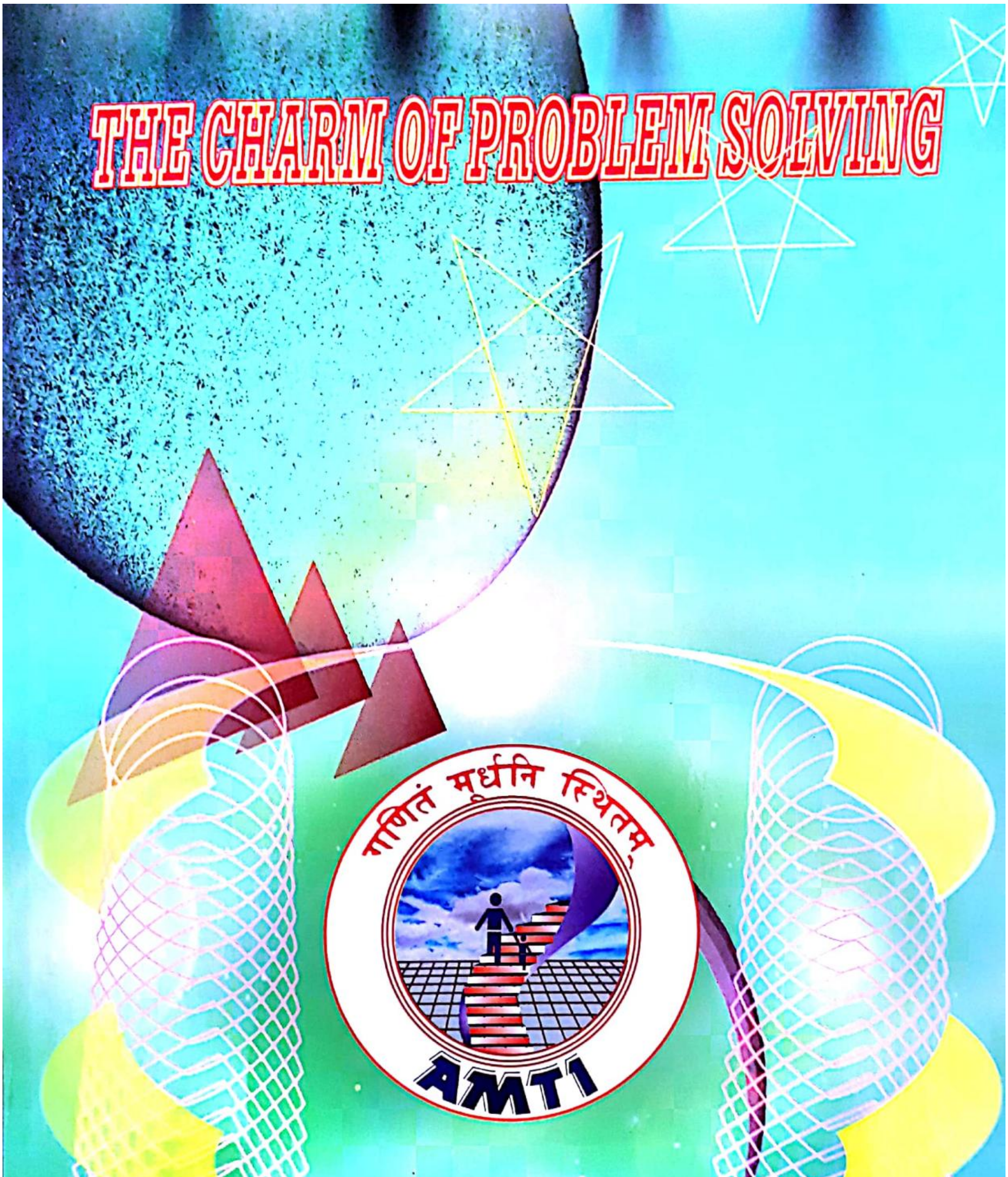
THE CHARM OF PROBLEM SOLVING
गणितं मूर्धनि स्थितम्
AMTI
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The AMTI

THE CHARM OF PROBLEM SOLVING

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BETWEEN US

Dear reader,

This book, a reference book for Primary onwards, is now in your hands as second and revised edition of our earlier one with this title. The AMTI has been conducting the National Talents Examinations in Mathematics for the past 37 years. For the past few years when the message was disseminated through the news media several parents and children contacted us and motivated us to include classes below IX also for this exercise.

So it was decided to include Classes IV to VI as Primary from 36th examination. The Primary students being completely new to this pattern, a reference material was found imperative for them. *KumariR. Vijayalakshmi,* our senior member, took up this work seriously. I take this opportunity to express my grateful thanks to her, and appreciation for this work, completed on time. I also thank Prof. *Smt. HemalathaThyagarajan, NIT, Trichy,* for having gone through the manuscript and helped in typesetting, proof reading and enriching the content too.

We hope and trust that the children will get the necessary lead they need from this book and increase the population of mathematically talented in our society. The author has tried to keep a conversational style as if facing primary children to present the content, which may be useful for higher levels also.

It is my duty to place on record our thanks to *Sri Donald Knuth* for technical assistance, *M.K. Graphics* and '*A Golden Horse*' for bringing out this book on time through the printing process.

Looking forward to your encouraging response and feed back, and with kind regards / best wishes,

22.06.2006

Yours sincerely,

(M.MAHADEVAN)
Gen. Secretary

Contents

Part I

Questions

Chapter 1

Problems on Number Properties

(1). Write all pairs of two digit numbers using the digits 2, 4, 6 and 8 so that each number has distinct digits. Which pairs add up to give the maximum sum? Which pairs have the maximum difference? Which pairs have the minimum difference?

(2). Solve the above problem using four
(a) Consecutive digits (b) Consecutive Odd digits.

(3). Find all pairs of three digit numbers using the digits 1, 2, 3, 4, 5, 6. Repetition of digits not allowed. Among these pairs, which pairs give the maximum minimum sum?
Which pairs give the maximum minimum difference?

(4). Find the number of pairs of four digit numbers using the digits 2, 3, 4, 5, 6, 7, 8, 9. Repetition of digits not allowed. Find the pairs whose
(a) sum is the maximum minimum
(b) difference is the maximum minimum

(5). Define a' (read post prime or successor of a) as $a+1$ and $'a$ (read preprime of a or predecessor of a) as $a-1$.
Evaluate the following:

(a) $(1'-'1)+(2'-'2)+(3'-'3)+\cdots+(100'-'100)$.

(b) $1'+2'+3'+\cdots+100'$.

(c) $'1+'2+'3+\cdots+'100$.

(d) $1'+3'+5'+\cdots+99'$.

(e) $2'+4'+6'+\cdots+100'$.

(f) $1'-'2+3'-'4\cdots+99'-'100$.

(g) $(1'+'1)'+(2'-'2)'+(3'+'3)'+\cdots(99'+'99)'+(100'-'100)'$

(h) $(1'+'1)'+'(2'+'2)+(3'+'3)'\cdots+(99'+'99)'+'(100'+'100)$

(i) $(1'-'1)'+'(2'-'2)+(3'-'3)'+\cdots+(99'-'99)'+'(100'-'100)$.

(j) $(1'-'1)'-'(2'-'2)+(3'-'3)'-'(4'-'4)\cdots+(99'-'99)'-'(100'-'100)$.

(k) $(1'+2')'-'(2'+3')+(3'+4')'-'(4'+5')\cdots+(99'+100')'-'(100'+101')$.

(l) $(1'+2')'+'(2'+'3)+(3'+'4)'+'(4'+5')\cdots+(99'+100')'+'(100'+101')$.

(6). Prove that $[a'+(a')']'-'[(a')'+((a')')']=0$.

(7). Verify

(a) $(1'+2'+3')=3\cdot 2'$.

(b) $(2'+3'+4')=3\cdot 3'$.

(c) $(3'+4'+5')=3\cdot 4'$.

Find the value of $(99'+100'+101')$.

(8). Show that $a+((a')')'=a'+(a')'$.

(9). Find the value of
$(a_1' - 'a_1) + (a_2' - 'a_2) + (a_3' - 'a_3) + \cdots + (a_{100}' - 'a_{100})$.

(10). Using the following formula,

$$a(b') = a(b+1) = ab + a$$
$$a'(b) = (a+1)b = ab + b$$
$$a('b) = a(b-1) = ab - a$$
$$('a)b = ab - b$$

write the numbers (a) 25 (b) 36 (c) 27 in the form $a(b') = ab + a$, $a('b) = ab - a$ in at least two different ways.
Eg.:

$$32 = 30 + 2 \quad = (2 \times 15) + 2 \quad = 2[15 + 1] \quad = 2(15')$$
$$32 = 34 - 2 \quad = (2 \times 17) - 2 \quad = 2[17 - 1] \quad = 2('17)$$
$$32 = 8 \times 4 \quad = 8 \times (3') \quad = 8 \times {}'5$$

and so on. In how many ways can a prime number be written in the above form?

(11). Using the four digits 2, 4, 6, 8 make a pair of two digit numbers with distinct digits, whose product is the maximum minimum.

(12). Given the numbers 4, 5, 6, 7, 8, 9

(a) Find the number of pairs of 3 digit numbers.

(b) Find all the pairs of numbers so that the sums of numbers in each pair are the same

(c) Find the product of the numbers in the above pairs.

(d) Which pairs have the maximum product?

(13). Some rectangles with integer sides have equal perimeter. Find all rectangles with perimeter (a) 36 cm (b) 64 cm (c) 100 cm. Find in each case, the sides of the rectangles with maximum area. (*Note:* all squares are rectangles.)

(14). It is given that the area of rectangles with integer sides are (a) 16 cm^2 (b) 144 cm^2 (c) 196 cm^2. Find the perimeters of all possible rectangles in each case. Find the sides of the rectangles with minimum perimeter.

(15). How many addition/subtraction problems can be constructed using the digits 2, 3, 4, 5, 6 and forming two numbers, one with three digits and the other with two digits? What is the maximum sum? What is the minimum difference?

(16). Find two three digit number pairs, using the digits 2, 4, 6, 7, 8 and 9, so that (a) the sum is maximum (b) the difference is minimum (all the six digits 2, 4, 6, 7, 8 and 9 must be used in the two numbers).

(17). What is the maximum minimum carry over when (a) 9 (b) 99 (c) 999 (d) 9999 single digit numbers are added.

(18). Find four single digit numbers such that when they are added, the carry over is 3. How many such answers can be found if (a) there is no repetition of numbers (b) repetition of numbers is allowed?

(19). Find any five single digit numbers which give a carry over 3, when added? How many such answers can be found if all the numbers are distinct?

(20). What is the maximum minimum carry over when (a) 100 (b) 1000 (c) 10,000 (d) 5253 single digit numbers are added?

(21). Ten non zero single digit numbers are added to get a carry over of (a) 8 (b) 5. What is the maximum number nines (9s) that can be used, out of the ten numbers? What are the other numbers?

(22). If 9 single digit numbers, each of which is greater than 4, and less than 9 are added, in how many ways can you get the sum (a) 50 (b) 52?

(23). (a) What is the maximum minimum carry over when two single digit numbers are multiplied?

(b) Find all possible digits in the tens place of the product of four distinct non zero single digit numbers.

(24). Four single digit numbers greater than one are multiplied and the digits in the 100s and 10s places are the same. Find such numbers if (a) repetition is not allowed (b) repetition is allowed.

Digital sum and Digital root: Sum of the digits of a number is called its digital sum. If the digital sum itself is a number with more than one digit, the digital sum of these digital sums are found till a *single digit digital sum* is obtained. This last number is called the *digital root* of the given number.

$n = 135893$
The digital sum of $n = n_1 = 1+3+5+8+9+3 = 29$.
The digital sum of $n_1 = n_2 = 2+9 = 11$.
The digital sum of $n_2 = n_3 = 1+1 = 2$, a single digit number.
This single digit number is the digital root 135893.

Digital root of n = remainder when the digital sum of n is divided by 9, if the remainder is non zero, or equal to 9, if the digital sum is divisible by 9.

(25). Find the digital sum and digital root of
(a) 123 456 789 (b) 246 868 42 (c) 135 792 468 135 79

(26). How many numbers, whose digits are greater than 2 and less than 8 and where each digit is used exactly once, have a digital sum 12? How many of these are divisible by (a) 3, (b) 6, (c) 4, (d) 8, (e) 24, (f) 5, (g) 15, (h) 25. Find the sum of all these numbers. What is the digital sum and digital root of the sum?

(27). What is the minimum number of digits of a number whose digital sum is (a) 99 (b) 88 (c) 55 (d) 100 (e) 1000 (f) 10, 000.

(28). How many five digit numbers have the digital sum (a) 2 (b) 3 (c) 4? Do you observe any pattern?

(29). The digital sum of a 1000 digit number is greater than 1200 and less than 1210. Find the maximum number of times the digit 4 can appear in the number. What can be the other non zero digits in the number?

(30). How many three digit numbers can be formed using the digits 2, 5 using each digit at least once? What is the sum of all these numbers? Find all the divisors of this sum.

(31). Find all the four digit numbers formed using the digits 2 and 5, using each digit twice. Find the sum of all the these numbers and divisors of this sum.

(32). How many four digit numbers can be formed using each of the digits (a) 0, 5 twice? (b) using one five and three zeros (c) one zero and three fives? What is the sum of these numbers in each case?

(33). If all the ten digits, zero to nine are used to form a ten digit number, (a) What is the smallest number? What is the biggest number? (b) What are their digital sums and digital root (c) What is the sum and difference of these numbers, and their respective digital roots?

(34). Find the tens place of the numbers $A \times B + A + B$, where A and B are non zero, non unit numbers.
Hint: Draw tabular column as follows.

S.No	A	B	$A \times B + A + B$	Ten's Digit
1	2	2	8	0
2	2	3	11	1
3	2	4	14	1
	⋮	⋮	⋮	⋮
	5	9	59	5
	⋮	⋮	⋮	⋮

(35). What are the carry overs when non zero, non unit, single digit numbers are (a) squared (b) cubed.

(36). Find the maximum and minimum carry over when two single digit numbers are multiplied. How many pairs of these numbers give these carry overs, if digits are (a) repeated (b) not repeated. Prepare a tabular column as shown below:

S.No.	Carry overs if digits are not repeated	No. of pairs	Carry overs if digits are repeated	No. of pairs
1.	0	11 pairs ($1 \times 2 \cdots 1 \times 9$, $2 \times 3, 2 \times 4$)	0	13 pairs
2	1	8 ($2 \times 5, \cdots 2 \times 9$ $3 \times 4, 3 \times 5, 3 \times 5$)	1 (4×4)	9
3	2	?	2	?
4	3	?	3	?
⋮	⋮		⋮	
9	8		8	

(37). Complete the division problem, by filling in the boxes.

(a)

```
              □ □ 9   Quotient
       ___________
 □ □ | 5 0 4 4 8
       □ □ □ ↓ ↓
       ---------
           9 4 8
           □ □ □
       ---------
             5 7   Remainder
```

(b)

```
       2 □ □      Quotient
       _________
 □ □ | 9 5 7 6
       □ □ ↓ ↓
       -------
         □ 7 6
         □ □ □
       -------
             8   Remainder
```

(38). Here is an addition problem. In this problem a, b, c are three single digit consecutive numbers. In the third row, the same three digits appear in some order. Find a, b, and c.

$$\begin{array}{|cccc|} \hline & a & b & c \\ & c & b & a \\ & \square & \square & \square \\ \hline 1 & 2 & 4 & 2 \\ \hline \end{array}$$

(39). Let the letters a, b, c, d represent four consecutive single digit natural numbers, $a > b > c > d$. The following is a four digit addition problem, where the boxes are also filled suitably by a, b, c and d in some order. Find the values of a, b, c and d.

$$\begin{array}{|ccccc|} \hline & a & b & c & d \\ & d & c & b & a \\ & \square & \square & \square & \square \\ \hline 2 & 2 & 1 & 0 & 1 \\ \hline \end{array}$$

(40). In the above problem for the same values of a, b, c and d, how many different answers can be got by rearranging the third row of the numbers alone?

(41). Find the highest power of 2 dividing $n = 36 \times 48 \times 26 \times 23 \times 18 \times 225$. What is the highest power of (a) 3 (b) 6 dividing n?

(42). What is the highest power of 10 dividing $n = 2^8 \times 5^9$ (b) $2^{16} \times 5^{13}$ (c) $4^{14} \times 5^{28}$? In each case find the number of zeroes at the end of the products.

(43). Define $n! = 1 \times 2 \times 3 \times 4 \times \ldots 10 \times 11 \times \ldots n$, *i.e.*, $n!$ is the product of all natural numbers from 1 to n.
eg: $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$.
$12! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12$

(a) What is the highest power of 5 dividing
(i) 10! (ii) 100!
(b) What is highest power of 2 dividing (i) 10! (ii) 100!
(c) What is the highest power of 10 dividing
(i) 10! (ii) 100!
(d) What is the number of zeroes at the end of (i) 10! (ii) 100!

(44). What is the (a) unit (b) tens digit of (i) 10! (ii) 100!

(45). Find n, such that $3! \times 5! \times 7! = n!$

(46). Evaluate $\dfrac{1^2 \times 2^2 \times 3^2 \times \ldots \times 98^2 \times 99^2}{(100!)^2}$

(47). Show that

$$\frac{n!}{(n-1)!} + \frac{(n-1)!}{(n-2)!} + \frac{(n-2)!}{(n-3)!} + \cdots + \frac{1!}{(0!)} = \frac{(n+1)!}{(n-1)! \times 2!}$$

(48). A five digit number n has the following property. The unit place represents the number of 4s in n, the tens place, the number of 3s, hundreds place, the number of 2s, the thousands place the number of zeros (in n). Find all such numbers n.

(49). It is a well known fact that Mahatma Gandhi was the man responsible for getting us the freedom. We won independence in 1947. Mahatma was born in 1869, so let us dedicate this to the memory of father of our nation. Find the smallest number by which
(a) 1869 should be multiplied so that the last four digits from the right is 1947 (i.e the product ends with 1947).
(b) 1947 should be multiplied so that the last four digits are 1869.

(50). Here is a problem in memory of Chacha Nehru, first prime minister of our country. What is the smallest number by which 1889 (his birth year) is to be multiplied so that the product may end in (a) 1947 (b) 1869.

(51). Using non zero single digit numbers, construct addition problems with
(a) two 2 digits addends
(b) three 2 digits addends
(c) two 3 digits addends
(d) two 4 digits addends,
so that the sum may be repeated digits not used in addends and no digit of the addends are repeated.

Example 1.	Example 2.
$\begin{array}{r} 35 \\ +\,42 \\ \hline 77 \\ \hline \end{array}$	$\begin{array}{r} 752 \\ +\,136 \\ \hline 888 \\ \hline \end{array}$

(52). Find how many two 2 digits addends give the sum (a) 44 (b) 55 (c) 66 (d) 77 (e) 88 (f) 99.

(53). My grandfather told me that in the year 2000, the last two digits of his year of birth was my father's age and the last two digits of my father's year of birth was his (grandfather's) age and that, both their ages (in 2000) were square numbers. Can you help me in finding the years of birth of my father and grandfather?

(54). Using the two single digit numbers 2 and 5, each at least once and using addition and/or subtraction write the numbers from 1 to 10 (It is desirable to use a minimum number of addition and/or subtraction).
Example:

$$11 = 5 + 5 + 5 - 2 - 2$$

$$= 2 + 2 + 2 + 2 + 2 + 5 - 2 - 2$$
$$= 5 + 2 + 2 + 2$$

Here the last representation of 11 is more economical.

(55). Write 1 to 10 using 3 and 5, each at least once, and using addition and/or subtraction.

(56). Choosing any two even numbers, can you write all natural numbers using addition and subtraction? If not which numbers can not be written?

(57). Can you write all numbers using 6 and 9? Why? Which numbers can be written? What is the smallest number that could be written? What relation does this smallest number have with 6 and 9?

(58). Repeat Qn. No. (57), replacing 6 and 9 by 25 and 15.

(59). The sum of three two digit numbers is 200. If each of the digits of the addends are replaced by the corresponding (a) 10's complement (b) 9's complement, what would be the sum of the three numbers.
Note: 5 is 9's complement of 4; 7 is the 10's complement of 3. If $a + b = 9$, then each is the 9's complement of the other, If $a + b = 10$ then each is the 10's complement of the other. For 10's complement no digit in the addends must be zero.

(60). If the sum of three, 3 digit numbers is 1475, find the sum of the numbers when each of the digits is replaced by (a) 9's complement (b) 10's complement.

(61). Find the number of 3 digit numbers, such that the first 2 digits are in ascending order and the last two digits are in descending order.

(62). Show that the highest power of two, dividing the product of the sum and difference of two odd natural numbers is 3.

(63). Find the number of n's such that $n = a^3b^2c$ where a, b, c are any three of the first four prime numbers. What is the least value of n? What is the biggest value of n.

(64). (a) Evaluate $1 - \left(\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \cdots + \frac{1}{999\times1000}\right)$
(b) If $1 - \left(\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)}\right) = 0.000001$, find n.

(65). Unit fractions are fractions with numerator 1. Express the following as indicated.

(a) $\frac{1}{2}$ as a sum of two distinct unit fractions.
(b) $\frac{1}{3}$ as a sum of three distinct unit fractions.
(c) $\frac{1}{4}$ as a sum of four distinct unit fractions.
(d) $\frac{1}{5}$ as a sum of five distinct unit fractions.
(e) $\frac{1}{6}$ as a sum of six distinct unit fractions.

(66). Write $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ and $\frac{1}{6}$ as difference of two unit fractions.

Chapter 2

Divisors and Multiples: lcm and gcd

(1). Find the number of divisors, sum of the divisors and sum of the reciprocals of the divisors of 6, 15, 35, 28, and 496.

(2). Find the number of divisors and the sum of the divisors of the squares 4^2, 3^2, 6^2, 10^2, 15^2. Deduce the property of the number of divisors of square numbers.

Perfect numbers: In Qn. No. 1, the sum of divisors of 6, 28 and 496 are found to be 12, 56 and 992 respectively. For each of these numbers the sum of the divisors was equal to twice the number. These numbers are called perfect numbers. These numbers are of the form $2^{n-1}(2^n - 1)$, where $(2^n - 1)$ is a prime number. [Proof of this will be provided later.] The sum of the reciprocals of the divisors of a perfect number is always 2.

(3). Prove that if any number which has more than 1 digit is divisible by 11, then the number got by reversing the digit is also divisible by 11.

(4). If the number $4a3b$ is divisible by 11, find all the values of a and b. Using the values of a and b and five other digits, construct at least 5, seven digit numbers divisible by 11, where digits cannot be repeated.

(5). Find all two digit numbers which are divisible by their (a) unit digit (b) tens digit (c) both unit and tens digit.

(6). Find all two digit numbers which are divisible by
(a) sum of their digits (b) product of their digits

(7). A two digit number is called a premium number, if it has both its digits prime. Find all premium numbers. If a premium number itself is a prime, it is called a prime premium number. Find all the 2 digit prime premium numbers.

(8). Can the sum of two prime numbers be a prime number? If yes, find all two digit prime numbers which are the sum of two prime numbers.

(9). Find the number of ways in which 100 can be written as the sum of two prime numbers.

(10). If $47\square \times 3\square 2 = 162792$, find the numbers to be placed in the boxes.

(11). A boy divided a certain number by 75 instead of by 72 and got both quotient and remainder to be 72. What should be the quotient and remainder if it is divided by 72.

(12). A boy divided a certain number by 9787, instead of 9778 and got the quotient 9778 and remainder 9782. Find the correct quotient and remainder.

(13). A student mathematician wanted to find a short cut for division by numbers nearer to 100, 1000, 10,000 etc. In one of the problems he divided and the number by the number 10,000 instead of by 9997 and the quotient was 979 and remainder 7067. How should he proceed to get the correct result (i.e when the number is divided by 9997)

(14). Find the highest power of six dividing the number

$$72 \times 727 \times 7272 \times 72727 \times 727272$$
$$\times 7272727 \times 72727272 \times 727272727?$$

(15). What is the highest power of 28 dividing the number $7 \times 14 \times 21 \times 28 \times 35 \times 42 \times 49 \times 56 \times 63 \times 70 \times 77 \times 84 \times 91 \times 98$?

(16). Define $n! = 1 \times 2 \times 3 \times \ldots \times (n-1)n$. $1! = 1, 0! = 1$. Find the highest power of 2, dividing 100! (For factorial numbers refer chapter 1)

(17). Find the highest power of 16 dividing 100!

(18). Find the highest powers of (a) 5 (b) 25 dividing 100!

(19). Using the result of problem 16 and 18 can you find the number of zeros at the end of 100!?
[*Hint:* Find the highest power of $10 = 2 \times 5$ dividing 100!. The highest power of 2 dividing 100! is bigger than the highest power of 5, dividing 100!. So, it is enough if you find the highest power 5 dividing 100!]

(20). What is the units place of $1! + 2! + 3! + \ldots + 100!$?

(21). What is the 10s place of $1! + 2! + 3! + \ldots + 100!$?

(22). *Note:* $n! = (n-1)! \times n$ Ex: $10! = 9! \times 10$.

(a) If $4 \times 5 \times 6 = n!$ find n.

(b) Express $4 \times 5 \times 6$ as $\frac{n!}{m!}$, where m and n are positive integers.

(c) Express 60 as $\frac{n!}{m!}$, where m and n are positive integers.

$$\left[\text{Hint: } 20 = 4 \times 5 = \frac{1 \times 2 \times 3 \times 4 \times 5}{1 \times 2 \times 3} = \frac{5!}{3!}\right]$$

(23). Rewrite the numbers in ascending order 75!
$2^{37} \times 3^{25} \times 5^{15} \times 7^{10} \times 11^{6}$, $13^{5} \times 17^{4} \times 19^{3} \times 23^{3} \times 29^{2} \times 31$

(24). If 5 divides a square number, then show that $5^2 = 25$ divides that square number.

(25). Find all possible remainders when a square number is divided by 3, 4, 5, 8 and 16. (Take the squares of 1, 2, 3, 4 and 5 or any other 5 consecutive number and check).

(26). Find all possible remainders when a cube is divided by 3, 4, 5, 8 and 16.

(27). Find all pairs (a, b) $10 \leq a, b \leq 20$, such that (a) 5 divides $a+b$, but neither a nor b. Also find the pairs with similar property for 7, 8 and 9.

(28). The product of two numbers a and b divides 48. a, b are not relatively prime to each other. Find all pairs (a, b), $1 < a \times b < 48$, $a \neq b$.

(29). 48 divides $a \times b$, 48 and a are relatively prime, Find all pairs (a, b) such that $10 \leq a \leq 30$, $10 \leq b \leq 100$.

(30). 12 divides $a \times b$, 12 does not divide a and neither does it divide b. If $1 < a < b$, find (a, b) such that (i) 12 and a (ii) 12 and b are relatively prime. Can a and b be relatively prime?

(31). Select a number which gives remainder 2, when divided by 3. Square and cube the number. Find the remainder when these are divided by 3. When an odd power of a number is divided by 3, what are the possible remainders? When even powers of a number is divided by 3, what are the possible remainders?

(32). Find all pairs (a, b), $100 \leq a, b \leq 200$, and such that their gcd is (a) 12, (b) 36, (c) 60 (d) 35.

(33). The gcd of 120 and a is 40 and $50 < a < 200$. Find all possible values of a.

(34). Find all pairs of numbers whose lcm is (a) 120 (b) 150 (c) 224.

(35). The lcm of a and 48 is 336. Find all values of a such that $20 \leq a \leq 50$.

(36). The lcm of a and 72 is 1800 and the gcd of b and 72 is 24. Find the lcm and gcd of a and b. Given that a is the biggest possible value less than 1800 and b is the smallest possible value greater than 24.

(37). A certain number gives a remainder 3 on dividing by both 15 and 24. Find the smallest such number.

(38). A certain number gives a remainder 5 on dividing by 7 and a remainder 7 on dividing by 15. Find the smallest such number.

(39). A certain number, on dividing by 6 gives a remainder 4. What are the possible remainders on dividing the same number by 36 $= 6^2$.

(40). 32 divides $n - 6$ and 64 and does not divide $n - 6$. Find the highest power of 2 dividing (a) $n + 2$ (b) $n + 6$

(41). k, m, n are natural numbers and 3^4 divides n, 3^5 does not divide n. 3^2 divides $(n-k)$, 3^3 does not divide $(n-k)$, 3^3 divides $(n+m)$, 3^4 does not divide $n+m$. Find the least possible values of k, n and m.

(42). Find all the four digit numbers that leaves a remainder 4 on dividing by 2005.

(43). Find all pairs of three digit numbers whose lcm is 2000. In each case find the gcd of the pairs.

(44). When five of the six numbers 2, 3, 5, 7, 9 and 11 are multiplied, the result is 2310. What will be the product if 5 is omitted and the other left out number is included for the multiplication.

(45). How many (a) 2 digit (b) 3 digit (c) 4 digit $\cdots$ n digit numbers can be got using the first four prime numbers as the only digits of the numbers. What is the sum of the numbers in each case?

(46). If $14{\star}{\star}5 \div 2{\star}3{\star} = 5$, $\star$ represent distinct digit in the division and all the digits from 1 to 9 are used only once. Replace the stars by appropriate digits.

(47). A rectangular box of measurements 20 cm $\times$ 16 cm $\times$ 12 cm is packed with identical cubes with edges of integral lengths. What is the (a) minimum (b) maximum number of cubes that could be packed. What is the length of each edge of the cubes in each case.

(48). A rectangular box of measurements 26 cm $\times$ 33 cm $\times$ 35 cm is packed tightly with smaller cardboard boxes having integer length of edges. In how many ways the bigger box can thus be packed? What are measurements of smaller boxes in each case.

(49). Given the numbers 2^2, 2^3, 2^4, 2^5, 2^6, 3^2, 3^3, 3^4 and 3^5. Find the number of distinct possible values that could be got by multiplying any two of the given numbers.

(50). Find all the possible integers less than 10,000 such that they are divisible by all the single digit positive integers.

Chapter 3

Sequences

Consider the following sequences:

(1). The sequence of natural numbers up to 100
i.e $S = 1, 2, 3, \ldots, 100$.
Divide each of these terms by (a) 2 (b) 3 (c) 4 (d) 5 (e) 8 (f) 16.
In each case find the sum of the remainders.

(2). $S = 1^2, 2^2, 3^2, \ldots, 100^2$.
Find the sum of the remainders on dividing each term by (a) 3 (b) 5 (c).4 (d) 8 (e) 16.

(3). $S = 1^3, 2^3, 3^3, \ldots, 100^3$
Find the sum of the remainders on dividing each term by (a) 3 (b) 5 (c) 4 (d) 8 (e) 16.

(4). Consider the following sequence of (triangular) numbers.

$$S_1 = \frac{1 \times 2}{2} = 1, \qquad S_2 = \frac{2 \times 3}{2} = 3,$$
$$S_3 = \frac{3 \times 4}{2} = 6, \qquad S_4 = \frac{4 \times 5}{2} = 10$$

Divide these terms up to 20 terms (20^{th} term is 210) by (a) 3 (b) 4 (c) 5 (d) 8 (e) 10. Find the sum of the remainders.
Hint: Write the sequence of remainders and find the sum of the sequence up to 20 terms.

(5). Construct the sequence following the instruction given: given the first two terms as 0, 1. the third term is got by adding second and the first terms. In general the n^{nd} term is $t_n = t_{n-1} + t_{n-2}$ *i.e.*, n^{th} terms is $(n-1)^{th}$ term $+(n-2)^{th}$ term. (The sequence thus obtained is called Fibonacii sequence.) Construct the sequence up to 10 terms.

(a) Find the sum of these ten terms.

(b) Find $5 \times t_1 + 8 \times t_2$.

(c) Multiply the result got in (b) by 11.

(d) Compare the 7^{th} term with the result got in (b).

(6). Starting with $t_1 = 2$. $t_2 = 5$, do as instructed in Problem (5) and answer all the Questions (a), (b), (c) and (d) of Problem 5.

(7). Repeat Problem 6 taking $t_1 = 5$ and $t_2 = 2$.

(8). Consider $f_0, f_1, f_2, \ldots, f_{19}$ of the first 20 terms of the Fibonacii sequence 0, 1, 1, 2, 3, 5, 8, ...Let $g_0, g_1, g_2, \ldots, g_{19}$ be the squared sequence; $g_n = f_n^2$. Now define a new sequence $t_1, t_2, \ldots, t_n$ where $t_n = g_n + g_{n-1}$. then show that $t_n = 2t_{n-1} + t_{n-2} + \cdots t_1$ for $n \geq 3$.
Also show that $t_1, t_1 + t_2, t_1 + t_2 + t_3, \ldots$ are also the alternate terms of the given Fibonacii sequence.

(9). Construct the sequence whose first two terms are 0, 1 and for $n \geq 3$, $t_n = t_{n-1} - t_{n-2}$. Find the sum of 100 terms of this sequence.
Hint: The first five terms are 0 1 1 0 -1.

(10). In a sequence $t_1 = a_1$, $t_2 = a_2$, $t_3 = t_2 - t_1$, $\ldots t_n = t_{n-1} - t_{n-2}$ for $n \geq 3$. Write the first 8 terms of these sequence. Find the sum to 5 terms, 6 terms and 100 terms.

(11). In a sequence $t_1 = 5$, $t_2 = 3$, $t_n = t_{n-1} - t_{n-2}$, $n \geq 3$. Find the sum to 100 terms. How does the answer change if $t_1 = 3$, $t_2 = 5$.

(12). Do the Problem 9 starting with (a) $t_1 = 1$, $t_2 = 2$, (b) $t_1 = 2$, $t_2 = 1$; (c)$t_1 = 89$, $t_2 = 55$ and (d)$t_1 = 55$, $t_2 = 89$.

(13). Consider the following sequence 1, 22, 333, ... where n^{th} term t_n consists n n's. Thus t_{20} is $\underbrace{202020\cdots20}_{\text{20 twenties}}$. Find the number of digits and the digital sum and digital roots of t_{244}. Find the number of the digits 2 appearing in t_{244}. Do you have yet another term having exactly the same number of twos as in t_{244}? How many times the digit 4 appear in t_{244}. Find another term where the same number of 4's appear.

(14). Find the digital sum of the 100^{th} term of the sequence defined in Problem (13).

(15). Consider the sequence defined in Problem (13). Construct a sequence whose terms are the number of digits in corresponding terms of the sequence in Problem (13).
Also construct a sequence whose terms represent the numbers which do not find a place in the above sequence (in ascending order).
Can you suggest a rule for the n^{th} term of these two sequences?

(16). Consider the sequence whose n^{th} term t_n is given by $\overline{n}\,\overline{n}\,\overline{n+1}$, where $\overline{n}$ is the digital representation of n. The first three terms are $t_1 = 112$, $t_2 = 223$ and $t_3 = 334$. t_{20} is 202021 and t_{500} is 500500501.

(a) Find the digital sum of t_{567}.

(b) Also find the sequence whose terms represent the number of digits in $t_1, t_2, t_3, \ldots t_n$ of the sequence defined above.

(c) Also find the sequence whose terms represent the numbers which do not occur in the above sequence.

(d) Find the sequence of numbers which are the digital sum of the above sequence $\overline{n}\,\overline{n}\,\overline{n+1}$.

(17). A sequence is defined using a single digit 2 or 1 or more digits 1, as follows:

$$\begin{array}{lll} t_1 = 12, & t_2 = 21, & t_3 = 112, \\ t_4 = 121, & t_5 = 211, & t_6 = 1112, \\ t_7 = 1121, & t_8 = 1211, & t_9 = 2111, \\ t_{10} = 11112. & & \end{array}$$

Find the 25^{th} term. How many terms will have the same number of ones as occurring in t_{28}? How many digits are used from t_1 to t_{28}? How many of these are 2's?

[*Hint:* t_1 and t_2 have 1 one, digits 3, 4, 5 have two 1's and so on. Thus the number of 1's increase in 1^{st}, 3^{rd}, 6^{th}, 10^{th} and so on. Do you remember these numbers which are of the form $\frac{n(n+1)}{n}$?]

(18). Let the sequence S, T, U and V be defined as $S = 12, 21, 2112, 211221, \ldots$.

S_n: the n^{th} term of s—the digits of S_{n-1} followed by the digits of S_{n-2} for $n \geq 3$.

T_n: the number of 1's in S_n,

U_n: the number of 2's in S_n,

V_n: the sum of the digits of S_n.

Continue the sequence S, T, U and V up to 20 terms. Observe sequences T and U. Find the sum of the digits of S_{10}. Find the sum to 10 terms of T,U and V. Find how many 1's and 2's are used to write 10 terms of S.

(19). Define S as follows: $S = 1, 2, 221, 2212212, \ldots$. $S_n =$ the digits of S_{n-1} repeated twice in that order followed by the digits of S_{n-2} for $n \geq 3$.
Write the first 10 terms. Generate 3 new sequences as in Question 18 and hence find the number of 2's and 1's used in the 15^{th} term of S by extending the terms of T, U and V. Also find the digital sum in the 15^{th} terms of S.

(20). Find the 2005^{th} term of the sequence
$S = 1, 23, 456, 789\,10, \ldots$
S_n has n natural numbers, with $\frac{n(n+1)}{2}$ as its last block of digits. Also find the number of digits, the digital sum and digital root of S_{2005}.

(21). In the sequence T, $t_{10} = 0$, $t_{11} = 1$, $t_n = t_{n-1} + t_{n-2}$. Find the first 9 terms of this sequence. Extend the sequence upto 20 terms. Find the sum of the first 20 terms.

(22). Generate the following sequence as instructed:
Starting with the natural number a, the second terms is $2a + 1$ if it is a prime number, if not, the biggest prime dividing $2a + 1$ is the second term. Likewise find the third, fourth, ... terms. The first few terms of S when $a = 2$ are, 2, $(2 \times 2 + 1) = 5$, $(2 \times 5 + 1) = 11$, $(2 \times 11 + 1) = 23$, $(2 \times 23 + 1) = 47$. $\because$ $2 \times 47 + 1 = 95 = 19 \times 5$, the 6^{th} term is 19. Find the sum to 100 terms.

(23). Generate the sequence P of prime numbers, starting with $t_1 = 3$. t_n is the biggest prime dividing $3t_{n-1} + 1$. The first 3 terms of P

are 3, 5, 2, ...

(a) Write the sequence P up to 10 terms.

(b) Find the sum to 100 terms of this sequence.

(24). Generate the sequence of numbers given $t_1 = 48$, $t_2 = 50$, $t_n = 2t_{n-1} - t_{n-2}, n \geq 3$. Find t_{50}, sum to 50 terms and sum to 100 terms.

(25). Starting the sequence S, with $t_1 = 6$, $t_2 = 9$ and $t_n = 2t_{n-1} - t_{n-2}$. Write the 100th term of the sequence. Which term of the sequence is the number 150? Find the sum of this sequence up to 2005 terms.

(26). Find the hundredth term of the sequence of triplets $(1,1,1)$, $1,2,3$, $(1,3,6)$, $(1,4,10)$, ... Suggest a general form (i.e t_n) of this sequence.

Check if $(1, 2005, 2005 \times 1003)$ is a term of this sequence. If yes, find the terms before and after this triplet.

(27). Consider the following sequences.

$S = 1, 2, 2^2, 2^3, \ldots 2^n$.

$T = 1, 3, 3^2, 3^3, \ldots 3^n$.

$U = 1, 5, 5^2, 5^3, \ldots 5^n$.

(a) Find the fiftieth term minus the forty ninth term of S, T, U

(b) Find the general form of $S_n - S_{n-1}$, $T_n - T_{n-1}$, $U_n - U_{n-1}$.

(c) In terms of the sequences S, T and U, what are the following sequences?

(i) 2, 6, 18, 54, ...

(ii) 4, 20, 100, 500, ...

(iii) 3, 12, 48, 192, ...

(iv) 5, 30, 180, 1080, ...

(28). $S = 1, (1 - \frac{1}{2}), (1 - \frac{1}{3}), (1 - \frac{1}{4}), \ldots (1 - \frac{1}{n-1}), (1 - \frac{1}{n})$. Find the product of 2005 terms.

(29). The n^{th} term S_n of a sequence S_n is $S_n = \dfrac{n(n+2)}{(n+1)^2}$. Write the first 10 term of this sequence. Now form a second sequence T as follows, $T_1 = S_1$, $T_2 = S_1 \times S_2$, $T_3 = S_1 \times S_2 \times S_3, \ldots$. Write down the first ten terms of the sequence T.
Form sequences U and V as $U_1 = T$, $U_2 = T_3$, $U_3 = T_5, \ldots U_n = T_{2n-1}$ and $V_1 = T_2$, $V_2 = T_4$, $V_3 = T_6, \ldots V_n = T_{2n}$. Give the general terms T_n, U_n and V_n. Show that each of the terms of U and V lies between $\frac{1}{2}$ and 1.

(30). Here are four sequences of fractions.

(a) $\dfrac{1}{96}, \dfrac{2}{95}, \dfrac{3}{94}, \dfrac{4}{93}, \dfrac{5}{92}, \ldots \dfrac{97}{1}$

(b) $\dfrac{1}{48}, \dfrac{2}{47}, \dfrac{3}{46}, \dfrac{4}{45}, \dfrac{5}{44}, \ldots \dfrac{48}{1}$

(c) $\dfrac{1}{97}, \dfrac{2}{96}, \dfrac{3}{95}, \dfrac{4}{94}, \ldots$

(d) $\dfrac{1}{99}, \dfrac{2}{98}, \dfrac{3}{97}, \dfrac{4}{96}, \ldots$.

Find the number of fractions in a, b, c and d which are irreducible? (i.e for which there is no common factors between the numerator and denominator).

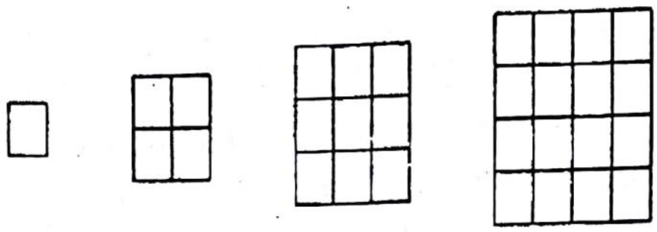
(31). Given the sequence 6, 10, 14, 15, 21, 22, 26. Find the next 7 terms.

[Hint: The terms of the sequence are numbers which are the product of 2 prime numbers in ascending order.]

(32). $S_n = 1-2+3-4+5-6\ldots$ up to n terms. Find $S_{2004}+S_{2005}+S_{2006}$.

[Hint: $S_1 = 1, S_1+S_2 = -1, S_1+S_2+S_3 = 2, S_1+S_2+S_3+S_4 = -2$]

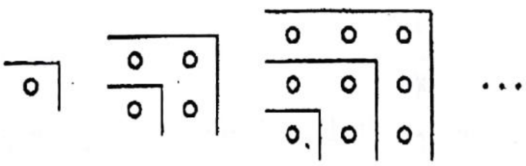
(33). Sequence of diagrams S_d are given here



Draw the next two diagrams of the sequence. What figure will represent t_n? We can derive a numerical sequence S from the sequences of the diagrams. S_n of the corresponding numerical sequence is, the total number of squares in the n^{th} diagram. Find S_5 of the numerical sequence.

(34). From the numerical sequence of Question (33), if a new sequence T is formed as follows. $S_1 = T_1$, $T_n = S_n - S_{n-1}$, $n \geq 2$. Find T_{15}

(35). Look at the sequence S_d of dot diagrams given below:



(a) Continue the diagram for three more terms.

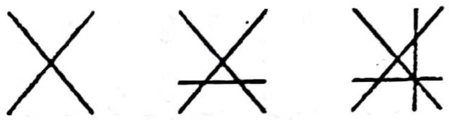
(b) Find the number of dots in 100^{th} diagram.

(c) Express the number of dots in the hundredth diagram as a sum of 100 numbers.

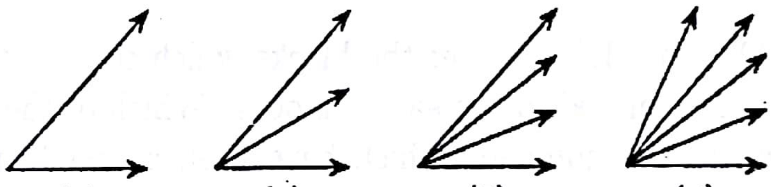
(36). Here is a sequence of segments with points on them. Find (draw) the 10^{th} term (10^{th} segments) of the sequence. How many segments are determined by the points on them?

 etc.

(37). Here is a sequence of intersecting lines, (where no three lines are concurrent). Draw the 4^{th} and 5^{th} terms of this sequence of lines. How many points are determined by 2005^{th} term (of lines)?



(38). Here is a sequence of angles determined by rays. Count the number of angles of each term of this sequence. Draw the next two terms. Find the total number of angles formed in the tenth term of the sequence



(1) (2) (3) (4)

(39). Define △a $= 2a$, $\boxed{a} = a^2$, ⌂a $= a^3$. Here is a sequence of diagrams with numbers in them:

$S =$ △1, $\boxed{2}$, ⌂3, △2, $\boxed{3}$, ⌂4, △3, $\boxed{4}$, ⌂5

The terms S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8 S_9. Extend the sequence upto S_{15}. Find which terms represent △10, $\boxed{10}$, ⌂10. Write the sequence T of numbers derived from S, upto 10 terms. The first three terms T_1, T_2 T_3 are 2, 4, 27. Which numbers repeat in the sequence T?

(40). Consider the following sequence:

$T = 1, 12, 123, 1234, \ldots$

$T_n = 1234 \cdots n$. (i.e t_n consists of digits of numbers from 1 to n).

(a) Find the number of 1s used in the sequence from 1 to 100 terms.

(b) Find the digital root of the sum of the digits from t_1 to t_{100}.

(41). Consider the following sequence:

$$\underbrace{\frac{1}{1}}, \underbrace{\frac{2}{1}, \frac{1}{2}}, \underbrace{\frac{3}{1}, \frac{2}{3}, \frac{1}{3}}, \underbrace{\frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}}, \dots$$

and the terms are grouped in blocks, the n^{th} block having n terms. Here the 5^{th} and the 8^{th} terms are in block 3 and 4 respectively. Name the block in which 50^{th} term lie? How many terms are there in the block in which 50^{th} term lies? What is the 50^{th} term?

(42). In Qn.No. (41), number the blocks which contain the fractions already counted in the earlier blocks. Which of the blocks have none of the numbers, which have been counted in the earlier blocks?

(43). The sequence of natural numbers are grouped as follows. $S = 1, \quad 2, 3, \quad 4, 5, 6, \quad 7, 8, 9, 10, \dots$. The n^{th} group of this sequence has n natural numbers. In which group does the natural number 100 lie? Find the 20^{th} group of numbers; Find the 30^{th} group of numbers. Can you generalise this to the n^{th} group of numbers?

(44). Find the sequence, T, whose terms are the sum of the numbers in the group of sequence of Qn. No. (40). Write the first 10 terms of this sequence and the sum to 10 terms.

(45). Consider the following sequence of number

$$\frac{2}{1}, \frac{5}{2}, \frac{10}{3}, \frac{17}{4}, \frac{26}{5}, \dots$$

Find the 100^{th} term of this sequence.

(46). Find another sequence whose terms are the difference of the consecutive terms of sequence S of Question (45).

Eg: $(\frac{5}{2}-\frac{2}{1}), (\frac{10}{3}-\frac{5}{2}), \cdots$ are the first two terms. Find the 100^{th} term of his sequence.

Chapter 4

Arithmetic of Remainders (Modulo Arithmetic)

(1). Write all two digit numbers that leave a remainder (a) 1 (b) 4 (c) 5 (d) 8, on dividing by 9

(i) In each case, how many numbers are odd? How many numbers are even?

(ii) What do you notice about the differences of any two numbers in each case.

(iii) Find the sum of any two numbers one from the answers of (a), the other from the answers of (d) and find the remainder on dividing this number by 9.

(iv) Repeat Question (iii) by choosing any two numbers, one from (b) and another from (c).

(2). (i) Write all two digit numbers that leave a remainder (a) 1 (b) 10 (c) 5 (d) 6 on dividing by 11.

(ii) What do you see about the differences of any two numbers in each case.

(iii) Show that the sums of the pair of numbers one from (a) and another from (b) is always divisible by 11.

(iv) Verify the same, choosing the two numbers one from (c) and another from (d).

(3). Define $a +_d b$ as the remainder obtained on dividing $a + b$ by d. Example: $132 +_{10} 73 = 5$.
Find (a) $18 +_7 42$ (b) $36 +_7 63$ (c) $(18 +_7 42) +_7 36$ (d) $18 +_7 (42 +_7 36)$.

(4). Define $a \times_d b$ as the remainder obtained on dividing $a \times b$ by d. Example: $52 \times_9 25 = 4$
Find (a) $34 \times_8 43$ (b) $17 \times_7 23$ (c) $64 \times_5 57$

(5). Using the definitions of $+_d$ and $\times_d$ in Questions (3) and (4) evaluate,

(a) $78 \times_{12} (42 +_{12} 37)$

(b) $(78 \times_{12} 42) +_{12} (78 \times_{12} 37)$

(6). *Modulo Arithmetic:* If two numbers a and b give the same remainder on dividing by d, we write
$a \equiv b \pmod d$ (Read a is congruent to b modulo d.) *Note:* if $a \equiv b \pmod d$, then $(a - b)$ or $(b - a)$ is divisible by d.
Example: $48 \equiv 57 \pmod 9$, $57 - 48 = 9$, is divisible by 9. Verify:
(a) $72 \equiv 1112 \pmod{10}$,
(b) $128 \equiv 18 \pmod{11}$,
(c) $144 \equiv 1112 \pmod{13}$.

Look back at Question (3):

$$\begin{aligned} 132 +_{10} 73 &= 5 \quad \text{is the same as} \\ 132 &\equiv 2 \pmod{10} \\ +\quad 73 &\equiv 3 \pmod{10} \quad \text{or} \end{aligned}$$

$$132 + 73 = 205 \equiv 5 \pmod{10}$$

Verify this result for Problem 4 and 5 also.

(7). Find x, if $5^6 \equiv x \pmod 7$, where x is a number less than 7.
Hint: $5^3 = 125 \equiv 6 \pmod 7$ and $5^6 = 5^3 \times 5^3$.

(8). Find the least value of x, for which

(a) $32^{12} \equiv x \pmod{13}$

(b) $8^4 \equiv x \pmod{15}$

(c) $455^{10} \equiv x \pmod{11}$

(9). Find the unit digit of (a) 2^{500} (b) 3^{500} and (c) 4324^{500}. (Unit digit of a number x is the digit y such that $x \equiv y \pmod{10}$. As $4^2 \equiv 6 \pmod{10}$, we get $4^4 \equiv 6 \times 6 \pmod{10} = 6 \pmod{10}$.)

(10). Find a two digit number n such that

a) $n \equiv 1 \pmod 4$

b) $n \equiv 1 \pmod 8$ and also

c) $n \equiv 1 \pmod{16}$

(*i.e.*, the same number n simultaneously leaves a remainder 1 when divided by 4, 8 and 16.)

(11). Find a two digit number which is simultaneously congruent to 1 (mod 5), (mod 7) and (mod 8).

(12). n is a two digit number simultaneously congruent to 1 (mod 5) and 3 (mod 4).

(a) Find all such n.

(b) For each n got in (a), find the least value of m such that

i. $n \equiv m \pmod{20}$,

ii. $n \equiv m \pmod{40}$.

(13). Find all $d > 1$ such that $100 \equiv 1 \pmod{d}$.

(14). Find the smallest natural numbers a and b such that $a \equiv 18 \pmod{7}$ and $b \equiv 42 \pmod{7}$. Hence find the smallest natural number c such that $c \equiv a + b \pmod{7}$.

(15). Fill up the addition $\pmod{7}$ table (*Note:* This is like $a +_7 b$).

+ (mod 7)	0	1	2	3	4	5	6
0							
1							
2							
3				6			
4					1		
5						3	
6							5

(16). Complete the following multiplication $\pmod{7}$.

× (mod 7)	0	1	2	3	4	5	6
0							
1							
2			4		1		
3							
4					2		
5							
6							1

(17). Using the addition $\pmod{7}$ and multiplication $\pmod{7}$ tables, solve

a) $x + 5 \equiv 3 \pmod 7$

b) $x \times 4 \equiv 1 \pmod 7$

c) $5 \times x + 3 \equiv 6 \pmod 7$

(18). Find $4 \times_7 (6 +_7 5) \pmod 7$ and $(4 \times_7 6) +_7 (4 \times_7 5) \pmod 7$.

(19). Complete the multiplication table $\pmod{36}$:

$\times_{36}$	4	8	16	20	28	32
4	16					
8		28				
16			4			
20				4		
28					28	
32						16

(20). Complete the multiplication table given. What do you get in 18-column and 18-row? Is it one of 9, 18 or 27?

$\times_{36}$	9	27	18
9			
27			
18			

(21). Using modulo arithmetic, show that
$127^{2005} + 721^{2005} + 217^{2005} + 272^{2005}$
is divisible by 7.

(22). Find the remainder when
$127^{2005} + 172^{2005} + 217^{2005} + 271^{2005} + 712^{2005} + 721^{2005}$
is divided by 7.

(23). Find the last two digits of
$2005^{2006} + 2006^{2005}$. (Hint: Take $\pmod{100}$.)

(24). Find the last two digits of
$2006 \times 2004 \times 1003 \times 3001 \times 4002 \times 6002.$

(25). Using modulo arithmetic show that
$23984^{15} + 119926^{25} + 359776^{35} + 71951^{45}$
is divisible by 117. $\left(\text{Hint: } 117 = 9 \times 13.\right)$

(26). Show that the sum of the squares of (a) 2 (b) 3 odd numbers can not be a square number.

Chapter 5

Pigeon Hole Principle

(1). A teacher asked a group of 10 students, to write a non zero single digit number. Prove that at least two students would have written the same number.

(2). Show that in a group of 13 boys, at least two of them would have their birth day in the same month.

(3). There are 8 cricket matches fixed between India A team and India B team. Show that at least two of the matches take place on the same day of the week.

(4). There are four pairs of socks of the same size and colour in a box. What is the least number of socks to be drawn from the box without seeing, so that one can wear a matched pair of socks?

(5). There are 5 pairs of yellow gloves and another 5 pairs of pink gloves in a box. They are all of the same size. What is the least number to be drawn so that one can wear a matched pair on both the hands?

(6). In a bag there are 10 blue balls and 7 red balls. What is the minimum number of balls to be drawn from the bag without seeing, so that there is at least

a) one red ball

b) one blue ball

c) two balls of the same colour

d) three balls of the same colour?

(7). A bag contains a large number of balls of three different colours. What is the least number of balls to be drawn so that there are at least (a) two balls (b) three balls of the same colour? What is the least number of balls to be drawn to get 10 balls of the same colour? (Can you recognise any pattern?)

(8). In the last decade in one February, 5 children were born on the same day of the week, but on different dates. Explain how this could happen? What are the different dates on which they were born? If the new year day that year was a Friday, on what day were the children born?

(9). A fifth standard student divided each of the dates in January by 7 and each of three of the remainders repeated 5 times. Explain, which other months of the year have this property? Can you find months in which

a) two of the remainders repeating 5 times

b) one of the remainders repeating 5 times

c) How many times the other remainders repeat?

(10). If seven numbers are selected at random from the numbers 5, 15, 25, 35, 45, 55, 65, 75, 85 and 95,

a) Show that there exist two numbers whose sum is 90.

b) Show that, if six numbers are chosen from out of these 10 numbers, then there exist two numbers, whose sum is 100.

c) Also show that in the former case, 6 numbers can be selected such that no pair gives a sum 90, and in the latter case, 5 numbers can be selected with no pair adding up to 100.

(11). During a particular week, a government hospital recorded the number of babies born to be 57 of which there were 12 twins and one triplets and one quadruplet. Prove that there is at least one day on which at least 9 babies were born.

(12). Ten packets of board pins, each containing 10 pins are bought for fixing the charts in an exhibition. These 10 packets respectively contain 4, 3, 1, 2, 3, 4, 3, 3, 2, 2 pins without the head. When 80 pins were chosen at random from these packets, show that there are at least 7 pins without head portions.

(13). 51 natural numbers are selected from 1 to 100. Show that there exist two numbers such that their total is 101.

(14). In the above problem, if 52 natural numbers are chosen, then show that there is a pair of natural numbers such that their sum is 103.

(15). If 48 numbers are selected at random from 1 to 100, then show that there exist at least two numbers such that their sum is divisible by 11.

(16). Prove that there exists a natural number whose digits consist of 1's and zeroes only and such that it is divisible by 2005.

(17). Prove that there exist a natural number whose digits consist of a's and zeroes only such that it is divisible by 2005, where $1 \leq a \leq 9$.

(18). Given the 5 integers a_1, a_2, a_3, a_4 and a_5, show that either, one of them is divisible by 5 or, a sum of several of them is divisible by 5.

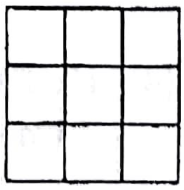
(19). Five points are plotted inside a circle. Show that there exist two points which form an acute angle with the centre of the circle.

(20). From the set of numbers 1, 2, 3, 4, 6, 9, 12, 18 and 36, if any six numbers are chosen then show that there exist at least two numbers whose product is 36.

(21). Show that in any group of 5 students, there are two students who have identical number of friends within the group.

(22). Prove that there exist two powers of 3 which differ by a multiple of 2005. (Hint: Look at the remainders when powers of 3 are divided by 2005.)

(23). There are 9 cells in a 3×3 square as shown here. When these cells are filled by the three numbers 1, 2 and 3, prove that, of the eight sums (along 3 rows, 3 columns and 2 diagonals), at least two sums are equal.



(24). 10 boys together gathered 40 shells. Show that at least two of them would have gathered the same number of shells.

(25). The digits 1, 2, 3, 4, 5 and 6 are divided into 3 groups. Show that the product of the numbers in at least one group must exceed 8.

Chapter 6

Algebra

(1). a and b are integers and $a^2 - b^2 = 17$. Find a and b. How many values do you get?

(2). a and b are positive integers and $a^2 - b^2 = 12$. Find a and b. How many values do you get?

(3). Find all integer solutions (x, y) for $4x - 5y = 9$.

(4). Find all natural numbers a and b satisfying
$7 + a + b = 10$.

(5). The subtraction problem shown alongside is carried out without borrowing. Find all the values of the digits a, b, c and d.

$$\begin{array}{rcc} & 4 & 5 \\ - & a & b \\ \hline & c & d \\ \hline \end{array}$$

(6). Find all values of the digits a, b, c, d, e, f, g and h, if the following (a) addition (b) subtraction problems have no carry over and no borrowing respectively:

(a)
$$\begin{array}{cccccc} & 6 & 5 & 7 & 8 \\ + & a & b & c & d \\ \hline & e & f & g & h \\ \hline \end{array}$$

(b)
$$\begin{array}{cccccc} & 6 & 5 & 7 & 8 \\ - & a & b & c & d \\ \hline & e & f & g & h \\ \hline \end{array}$$

(7). Find all possible values of the digit a such that the addition part of the multiplication problem has no carry over and hence find the product: $28 \times 3a =$ ____

(8). Find all values of the digits a and b such that 9 divides $ab5$. In each case find the quotient.

(9). a, b, c and d are four natural numbers such that

$$a+b+c=53, \qquad b+c+d=51$$
$$c+d+a=57, \qquad d+a+b=58.$$

Find the values of $a+b+c+d$, $a-b$, $b-c$, $c-d$ and $d-a$.

(10). Find all natural numbers n, such that $\dfrac{(n+2)(n+2)}{(n-3)}$ is a natural number.

(11). For all natural numbers x, show that 5 divides $x^2+(x+1)^2+(x+2)^2+(x+3)^2+(x+5)^2$.

(12). Show that the sum of the squares of 6 consecutive numbers decreased by 1 is always divisible by 6. (Hint: Write the consecutive numbers as a, $a+1$, $\ldots a+5$.)

(13). Show that $(x-3)^2+(x-2)^2+(x-1)^2+x^2+(x+1)^2+(x+2)^2+(x+3)^2$ is divisible by 7.

(14). Define $a \star b = 2a+2b-ab$. Find x, given $(4 \star x) \star 5 = 3 \star (5 \star x)$.

(15). Define $a \star b = a+b+1$. Find a and b if $(a \star 5) = (b \star 4) = 11$, and hence find $a \star b$ for the values you have found for a and b.

(16). It is given that $(a-b)^2+(b-2c)^2+(c-3d)^2+(d-4e)^2=0$ and $a+b+c+d+e=130$. Find the values of a, b, c, d and e. (Hint: Sum of the squares is zero!)

(17). Prove that $\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$. Use it to find $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\frac{1}{42}$.

(18). Find natural numbers a and b given $\frac{1}{3}+\frac{a}{b}=\frac{1+a}{4+b}$.

(19). Find n if,
$$\frac{x^6+x^6+x^6+x^6}{y^6+y^6+y^6}\times\frac{z^6+z^6+z^6+z^6+z^6+z^6}{k^6+k^6}=2^n,$$
where $x=2k$ and $y=\frac{1}{2}z$.

(20). Prove that the sum of the quotient and the remainder is always odd, when a two digit prime number is divided by another two digit prime number. In general, show that when any prime number is divided by another prime number then the sum of the quotient and the remainder is always odd.

(21). Define 9's complement of a single digit number a, denoted as a_{9c} as $9-a$ *i.e.*, $a_{9c}=9-a$.

a) Find a,b such that $(a+b)_{9c}=a_{9c}+b_{9c}$.

b) Find a such that $a-a_{9c}=17$.

(22). A two digit number is such that when the product of the digits is added to the sum of the digits, it gives the two digit number. Find the number.

(23). Find the minimum value of the expressions.

a) x^2+73, x is an integer, and

b) $x^2 - 6x + 12$, x is a natural number.

(24). Find the maximum value of the expressions:

a) $40 - y^2$, y is an integer

b) $33 + 10x - x^2$, x is an integer.

(25). Find the value of

a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

b) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$

c) $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \cdots$
(Hint: Group all the terms after 1 and take out common factor to get the original series.)

(26). Given $x^2 + x + 1 = 0$. Find the value of
(a) $x + \frac{1}{x}$ (b) $x^2 + \frac{1}{x^2}$ (c) $x^4 + \frac{1}{x^4}$ (d) $x^8 + \frac{1}{x^8}$

(27). Observe the pattern:

$$1 = 1^2$$
$$1 + 2 + 1 = 2^2$$
$$1 + 2 + 3 + 2 + 1 = 3^2$$
$$\cdots\cdots$$
$$1 + 2 + 3 + \cdots + n + (n-1) + (n-2) + \cdots + 1 = n^2$$

Using this prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

(28). Find n, given $(8^{73} + 2)^2 - (8^{73} - 2)^2 = 8^n$.

(29). If $a \times a + b \times b = c \times c$ and $p \times c = a \times b$,
prove that $\frac{1}{p \times p} = \frac{1}{a \times a} + \frac{1}{b \times b}$.

(30). Look at the number pattern given below:

$$1 = 1 \times 1 \times 1$$
$$3 + 5 = 2 \times 2 \times 2$$
$$7 + 9 + 11 = 3 \times 3 \times 3$$
$$13 + 15 + 17 + 19 = 4 \times 4 \times 4$$

Generalise and prove the algebraic relation for the pattern. (Hint: Write the n^{th} group on the LHS.)

(31). Observe the pattern and generalise for the n^{th} row of numbers:

A	B	C
$1, 2$	3	0
$1, 2, 3$	$4, 5$	3
$1, 2, 3, 4,$	$5, 6, 7$	8
$1, 2, 3, 4, 5$	$6, 7, 8, 9$	15
$\vdots$	$\vdots$	$\vdots$

(*Hint:* The numbers in column C is the difference between the sums of numbers in columns A and B. $(n-1)^{st}$ row of A has the first n natural numbers and B has $(n-1)$ natural numbers starting from $n+1$.)

32). Consider the ordered pairs (a, b), (c, d), where a, b, c, and d are whole numbers. We define $(a, b) = (c, d)$ if and only if $a + d = b + c$. Define the operation $\star$ on the ordered pairs given above $(a,\ b) \star (c,\ d) = (ac + bd,\ ad + bc)$
Example: $(2,\ 3) \star (10,\ 7) = (20 + 21,\ 30 + 14) = (41,\ 44)$.
Note that $(41,\ 44) = (1,\ 4)$.
Evaluate:

(a) (1) $(5,\ 4) \star (4,\ 5)$ (2) $(3,\ 4) \star (4{,}5)$ (3) $(6,\ 5) \star (6,\ 5)$ and express the result so that each term of the ordered pairs is the lowest possible.

(b) Find ordered pairs (a, b) such that
$(2, 3) \star (a, b) = (2, 3)$. Is $(2, 3) \star (4, 3) = (2, 3)$?
Is $(2, 3) \star (a+1, a) = (2, 3)$ for all natural numbers a?

(c) In the above ordered pairs, define
$(a, b) \odot (c, d) = (a+c, b+d)$.

(i) Find (x, y) such $(2, 3) \odot (x, y) = (2, 3)$, where x and y are natural numbers.

(ii) Is $(2, 3) \odot (a, a) = (2, 3)$ for all natural numbers a?
Show that $(3, 2) \star [(2, 3) \odot (3, 4)] = [(3, 2) \star (2, 3)] \odot [(3, 2) \star (3, 4)]$

(33). Consider the ordered pairs, (a, b), (c, d), a, b, c, d being whole numbers. $(a, b) = (c, d)$ if and only if $ad = bc$, $b, d \neq 0$.

(a) Find at least 5 ordered pairs equal to (i) $(2, 3)$, (ii) $(1, 1)$, (iii) $(0, 1)$.
Define $(a, b) \star (c, d) = (ad + bc, bd)$
Example: $(5, 4) \star (4, 5) = (25 + 16, 20) = (41, 20)$. $(41, 20) = (82, 40) = (123, 60)$ and so on.

(b) Find

(i)	$(1, 2) \star (3, 4)$,	(ii)	$(2, 1) \star (2, 1)$,
(iii)	$(4, 3) \star (3, 4)$,	(iv)	$(5, 4) \star (1, 1)$,
(v)	$(5, 4) \star (0, 1)$,	(vi)	$(5, 4) \star (0, 2)$,
(vii)	$(5, 4) \star (0, 3)$,	(viii)	$(5, 4) \star (0, 100)$.

(c) Define: $(a, b) \odot (c, d) = (ac, bd)$.
Example: $(2, 3) \odot (3, 4) = (6, 12) = (3, 6) = (4, 8) = (1, 2)$
Evaluate:

(i) $(4, 3) \odot [(5, 16) \star (7, 16)]$

(ii) $[(4, 3) \odot (5, 16)] \star [(4, 3) \odot (7, 16)] = (3, 4)$

(iii) $(3, 4) \odot [(2, 3) \star (2, 3)]$ and
$[(3, 4) \odot (2, 3)] \star [(3, 4) \odot (2, 3)]$.

(d) Find the ordered pairs (a, b) such that

(i) $(3, 4) \odot (a, b) = (3, 4)$

(ii) $(3, 4) \star (a, b) = (3, 4)$

(iii) $(3, 4) \odot (a, b) = (1, 1)$
(*Note:* $(1, 1) = (2, 2), (3, 3) = (4, 4) \ldots$)

(iv) $(3, 4) \star (a, b) = (0, 1)$

(34). Consider the ordered triplets $(a_1, a_2, a_3), (b_1, b_2, b_3) \ldots$ where $a_i, b_i \in Q, i = 1, 2, 3$. (Q is the set of all rational numbers.) Define $(a_1, a_2, a_3) \star (b_1, b_2, b_3)$, if and only if $a_i = b_i$, $i = 1, 2, 3$.
Definition 1: Define $\star$ on the triplets as follows:
$(a_1, a_2, a_3) \star (b_1, b_2, b_3) = (a_1 + b_1, a_2 \sim b_2, a_3 + b_3)$
($a_2 \sim b_2$– difference between a_2 and b_2).
Definition 2: Define $\odot$ on the triplets as follows
$(a_1, a_2, a_3) \odot (b_1, b_2, b_3) = (a_1 b_1, a_2 + b_2, a_3 b_3)$.
Evaluate:

(i) $(1, 2, 3) \star (3, 2, 1)$

(ii) $(1, 2, 3) \odot (3, 2, 1)$

(iii) Find (a, b, c) such that

$$(4, 5, 6) \star (a, b, c) = (4, 5, 6)$$
$$(4, 5, 6) \odot (a, b, c) = (4, 5, 6)$$

(iv) Evaluate:

(a) $(4, 5, 6) \odot ((2, 3, 4) \star (3, 4, 5))$

(b) $[(4, 5, 6) \odot (2, 3, 4)] \star [(4, 5, 6) \odot (3, 4, 5)]$.

(35). Observe the patterns in columns A, B and C, where $x \neq 1$.

Column A

$$(1 - x) \times 1 = (1 - x)$$

$$(1-x)(1+x) = 1-x^2$$
$$(1-x)(1+x+x^2) = 1-x^3$$
$$(1-x)(1+x+x^2+x^3) = 1-x^4$$
$$(1-x)(1+x+x^2+x^3+x^4) = 1-x^5$$
$$(1-x)(1+x+x^2+\cdots+x^{n-1}) = 1-x^n$$

Column B

$$(1+x)\times 1 = (1+x)$$
$$(1+x)(1-x+x^2) = 1+x^3$$
$$(1+x)(1-x+x^2-x^3+x^4) = 1+x^5$$
$$(1+x)(1-x+x^2-x^3+x^4-x^5+x^6) = 1+x^7$$
$$\vdots \qquad \vdots$$
$$(1+x)(1-x+x^2-x^3+\cdots-x^{2n-1}+x^{2n}) = 1+x^{2n+1}$$

Column C

$$(1-x)\times 1 = (1-x)$$
$$(1-x)(1+x) = (1-x^2)$$
$$(1-x^2)(1+x^2) = (1-x^4)$$
$$(1-x^4)(1+x^4) = (1-x^8)$$
$$\vdots \qquad \vdots$$
$$(1-x^n)(1+x^n) = (1-x^{2n})$$

Solve the following problems using the results given in columns A, B and C.
Write $1-x^{10}$ in two different ways as a product of factors and hence show that

(a) $(1+x)(1+x+x^2+x^3+x^4)(1-x+x^2-x^3+x^4) = (1+x+x^2+\cdots+x^9)$

(b) $(1+x+x^2+x^3+x^4)(1-x+x^2-x^3+x^4) = (1+x^2+x^4+x^6+x^8)$

(c) Prove that $(1+x+x^2)(1-x+x^2) = (1+x^2+x^4)$.

(36). Factorise $(1-x^8)$ in two ways and hence show that $(1+x)(1+x^2) = (1+x+x^2+x^3)$.

(37). Factorise $(1-x^6)$ in two ways and hence show that $(1+x+x^2)(1+x^3) = (1+x+x^2+x^3+x^4+x^5)$

(38). Find $(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})$

(39). Show that $(1-x^{32}) \div (1-x^2) = (1+x^2+x^4+\cdots+x^{30})$.

(40). Given

$$\underbrace{a+a+a+\cdots+a}_{x \text{ times } a} = a^2b$$

$$\underbrace{b+b+b+\cdots+b}_{y \text{ times } b} = ab^2$$

where a, b, x, y are natural numbers.

Find $(\underbrace{x+x+x+\cdots+x}_{y \text{ times}})(\underbrace{y+y+y+\cdots+y}_{x \text{ times}})$

in terms of a and b.

(41). Given $\frac{a}{c} = \frac{c}{d}$, show that $\frac{pa+qb}{ra+sb} = \frac{pc+qd}{rc+sd}$, where a, b, c, d, p, q, r, s are natural numbers.

(42). Given two consecutive numbers and their sum, show that exactly one of them is divisible by 3.

Pythagorean Triples: Any three natural numbers a, b, c $a < b < c$, and such that $c^2 = a^2 + b^2$, are called Pythagorean triples. In addition if a, b, c have no common divisor, then a, b, c are called Primitive Pythagorean Triples in short PPT.

If a triangle has for its sides, Pythagorean triples, then it is a right angled triangle. The biggest side, which is opposite the right angle is called the hypotenuse. In a Primitive Pythagorean

triples, the length of the hypotenuse is always an odd number and of the other two sides, one is odd and the other is even and they are called the odd leg and even leg respectively.

(43). Show that given two numbers a, b, one odd and the other even, and relatively prime, the numbers a^2+b^2, a^2-b^2, and $2ab$ can represent the sides of a right angled triangle.
[Hint: Show that $(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$]. The sides are relatively prime.

(44). Using the above formula, given the odd leg of a right angled triangle as 45 units, find the even leg and hypotenuse of the triangle. How many such triangles do you get? How many of them are primitive?
(Hint: $(a^2-b^2) = (a-b)(a+b) = 45 = 5 \times 9$ and a, b are integers, find a, b. Here they are 7 and 2 respectively. Find $2ab$ and a^2+b^2. You get sides of one right angled triangle. Similarly do for $a^2-b^2 = 1 \times 45$ and $a^2-b^2 = 3 \times 15$.)

(45). Find all the PPTs given the odd leg to 105 units.

(46) Show that $4984^{2005} + 1027^{2005} + 978^{2005} - 2979^{2005}$ is divisible by 2005. (Hint: $x^n - y^n$ is divisible by $x-y$ for all n and $x^n + y^n$ is divisible by $x+y$ for odd n.)

(47). Show that $1562^{2n+1} - 636^{2n+1} + 1646^{2n+1} - 567^{2n+1}$ is divisible by 2005 for all positive integer values of n.

(48). Given $ab = cd$, $a-b > c-d$, show that $a+b$ is greater than $c+d$ (a, b, c, d are positive integers $a > b$, $c > d$.)

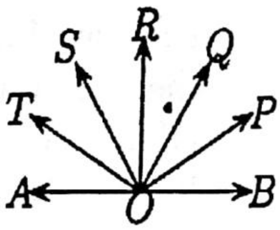
(49). Given $a+b = c+d$, $a-b > c-d$, show that $ab < cd$. (a, b, c, d are positive integers $a > b$, $c > d$.)

(50). Given $a+b > c+d$ and $(a-b) = (c-d)$, show that $ab > cd$, (a, b, c, d are positive integers, $a > b$, $c > d$).

CHAPTER 7

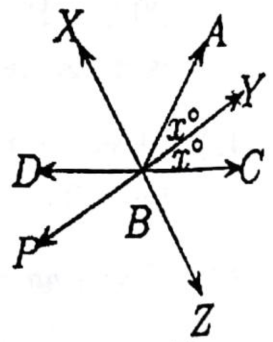
Geometry

(1). In the adjoining figure, AOB is a line and rays OP, OQ, OR, OS and OT are drawn so that
$\angle BOP = \angle POQ = \angle QOR$
$= \angle ROS = \angle SOT = \angle TOA$.



a) How many angles are formed, including the straight angle?
b) How many of them are acute?
c) How many are right angles?
d) How many are obtuse angles?

(2). In the adjoining figure, points D, B, C are collinear. XB is produced to Z and YB is produced to P. $\angle XBY = 90°$.
(a) Show that $\angle XBD = \angle XBA$
(b) Find $\angle DBZ$ and $\angle PBC$ in terms of $x°$.



(3). In a triangle each of the smaller angles is less than 45°. Show that the Δ is obtuse angled.

(4). In a triangle each of the smaller angles is greater than 45°. Show that the Δ is acute angled.

(5). In a triangle, each of the smaller angles is half the biggest angle. Find the angles of the triangle.

(6). In a triangle, each of the bigger angles is twice the third angle. Find the angles of the triangle.

(7). In an obtuse angled triangle, exactly one angle is twice one of the other two angles. If all the angles have integer values (in degree measure), find the number of such triangles and their angle measures. How many of them have the same obtuse angle?

(8). In a right angled Δ, one angle is twice the other angle. How many triangles can satisfy this condition? What are their angle measures?

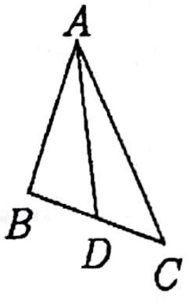
(9). In an acute angled scalene triangle, one angle is twice one of the other two angles. How many pairs of these triangles have angle measure of one angle common?

(10). The perimeter of a triangle with integer sides is (a) 18 cm (b) 13 cm (c) 8 cm. Find the number of triangles that can be constructed.

[*Note:* 1. Sum of the lengths of any two sides of a Δ is greater than the length of the third side.
Note: 2. This type of problems can also be considered as problems in number theory]

(11). $x > y > z$ are three positive integers. Show that a triangle can be constructed with sides of length (a) $x+y$, $y+z$, $z+x$ (b) x, $y+z$, $x+z$.

(12). ABC is a Δ. D is the mid-point of BC. Show that $\dfrac{AB + AC - BC}{2} < AD.$



(13). In Problem (12) above, if E and F are the midpoints AC and AB respectively, show that

(i) $BE > \dfrac{AB + BC - AC}{2}$

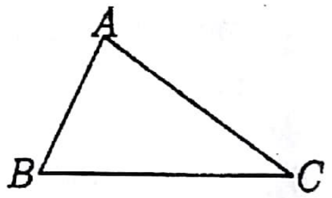
(ii) $CF > \dfrac{AC + BC - AB}{2}$

(*Note:* AD, BE and CF are the medians of the ΔABC.)

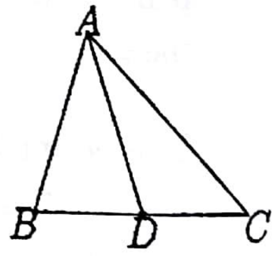
(14). Using the results of Problems 12 and 13, show that sum of the medians is greater than the semiperimeter of the triangle.

(15). In any triangle ABC, if $AB < AC$ then $\angle C < \angle B$. Prove.

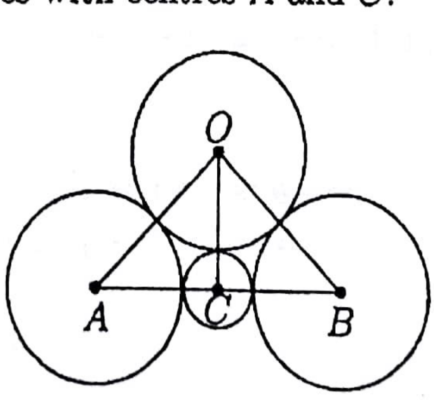
[Angle opposite the shorter side is less than the angle opposite the larger side]

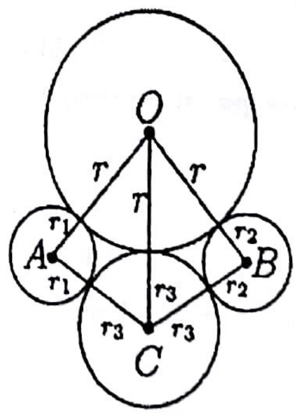


(16). In ΔABC, AD is the median (D is the mid point of BC). $AD > DC = BD$. Prove that $\angle BAC$ is acute.



(17). In the figures given below, the circle with centre O, touches the circles with centres A, B, C, and the circle centre B touches the circles with centres A and C.

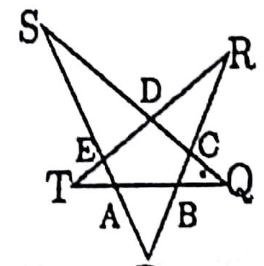




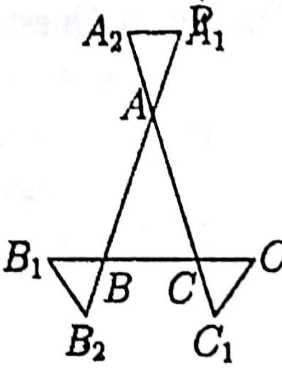
Show that if A, B, C are collinear, the radius of the circle centre O is greater than the radius of at least one of the circles with centres A, B, C. Will this be true when A, B, C are not collinear?

(18) $ABCDE$ is a pentagon and the star with five vertices $PQRST$ is drawn extending the sides of the pentagon. Calculate the sum of the angles of the star. $(\angle P+\angle Q+\angle R+\angle S+\angle T)$.

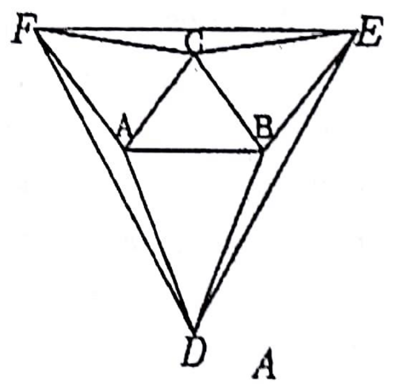
Note: Sum of all the angles of the pentagon is 540°.



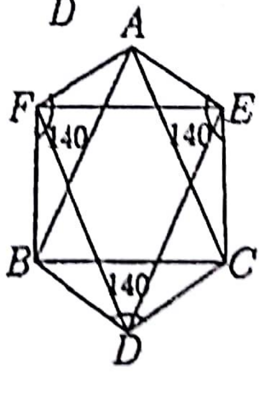
(19). ABC is a triangle and the sides are extended to get three more Δs at the three vertices as shown. Find the sum of the angles formed at A_1, A_2, B_1, B_2, C_1, C_2.



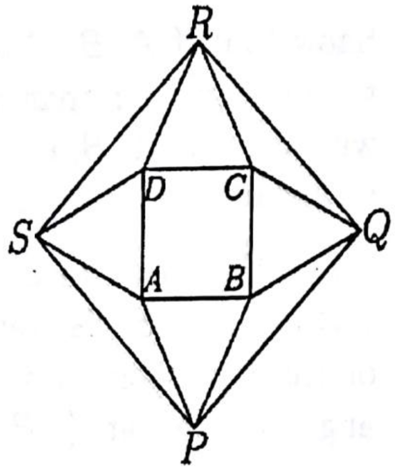
(20). ABC is an equilateral triangle. Isosceles triangle ABD, BCE and CAF are drawn, with $AD = BD = BE = CE = CF = AF$ and base angles of these Δs being 70° each. Find the angles of the triangle DEF and hence show that it is an equilateral triangle.



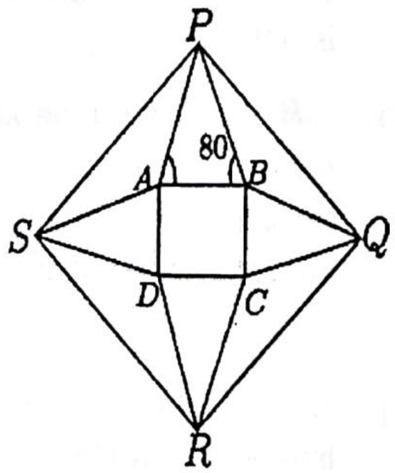
(21). ABC is an equilateral triangle. On the sides of this triangle isosceles Δs, with vertical angles 140° are drawn as shown in the figure. As in Question (20), the vertices DEF are joined. Show that DEF is an equilateral triangle.



(22). In the adjacent figure, on the sides of the square $ABCD$, equilateral Δs ABP, BCQ, CDR and DAS are drawn. Show that $PQRS$ is a square.

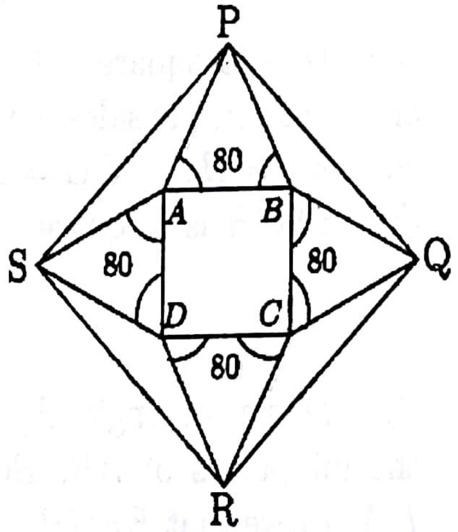


(23). As in Question 22, isosceles Δs, with base angles 80° are drawn on the sides AB, BC, CD and DA of the square $ABCD$. Show that $PQRS$ is square.



(24). On the sides of the square $ABCD$, if congruent isosceles Δs are drawn, then the vertices of these triangles form a square, whatever be the base angles. Prove. For what measures of base angles, do the sides of the square joining the vertices of the isosceles Δs pass through the vertices of the square $ABCD$?

(25). $ABCD$ is a rectangle. On the sides of this rectangle, equilateral triangles ABP, BCQ, CDR and ADS are constructed. Prove that $PQRS$ is a rhombus.



(26). In Question 25, instead of equilateral isosceles Δs of equal base angles $80°$ are constructed. Show the $PQRS$ is a rhombus.

(27). In the above question, whatever be the equal base angles, prove that the vertices $PQRS$ enclose a rhombus. Find for what values of the base angles, the sides of the rhombus

(a) will pass through vertices of the rectangle

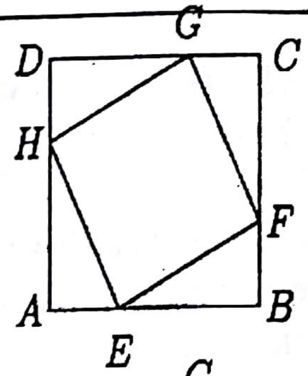
(b) will intersect the sides of the rectangle.

(28). If equilateral triangles are drawn on the sides of a regular pentagon(5 sided figure with sides and angles equal), show that the figure got by joining the vertices is also a regular pentagon.

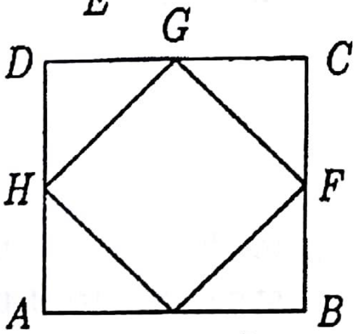
(29). In Question 28, instead of equilateral Δs, if isosceles triangles of equal base angles are drawn, verify if the figure obtained by joining the outer vertices of the Δs so constructed is a pentagon.

(30). Generalise Questions (28) and (29) to a regular hexagon, and prove. Investigate, corresponding to those for any regular polygon, the results stated in Questions (28) and (29).

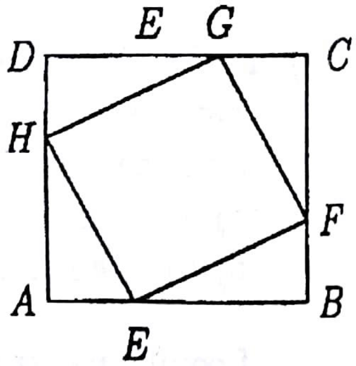
(31). $ABCD$ is a square. E, F, G, H are points on the sides of this square, with $AE = BF = CG = DH$. Show that $EFGH$ is a square.



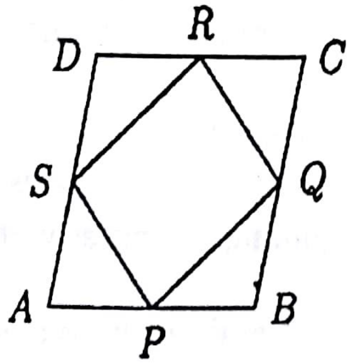
(32). $ABCD$ is a rectangle; E, F, G, H are the midpoints of AB, BC, CD and DA. Prove that $EFGH$ is a rhombus.



(33). $ABCD$ is a rectangle, with AB =8 and BC =6.4 units. E, F, G, H are points on the sides AB, BC, CD and DA such that $AE = CG = \frac{1}{4}AB = \frac{1}{4}CD$, $BF = DH = \frac{1}{4}CB = \frac{1}{4}AD$. Show that $EFGH$ is a parallelogram.

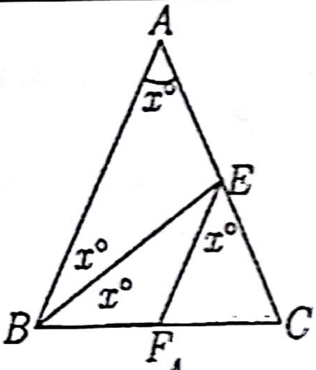


(34). P, Q, R and S are the mid points of the sides AB, BC, CD and DA respectively of the rhombus $ABCD$. Prove that $PQRS$ is a rectangle.

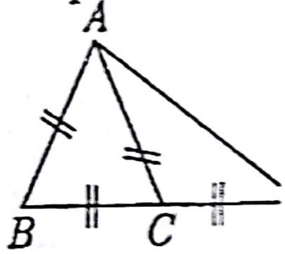


(35). In the above question if $ABCD$ is a parallelogram, prove that $PQRS$ is also a parallelogram.

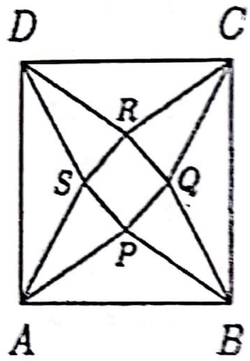
(36). In the adjoining figure, $EF = EC$. Find the value of x and show that ΔABC is isosceles.



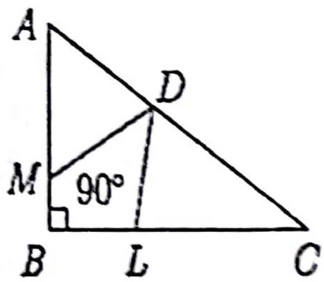
(37). In the adjoining figure $AB = AC = BC = CD$. Find the angles of the Δs ABC, ACD and BAD.



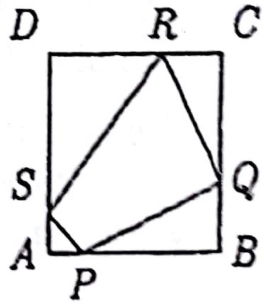
(38). $ABCD$ is a square; (AS, AP), (BP, BQ), (CQ, CR) and (DR, DS) are the four pairs of angle trisectors of the angles A, B, C and D. The angle trisectors of adjacent angles intersect at P, Q, R and S as shown here. Show that $PQRS$ is a square. (Trisect—divide into 3 equal parts.)



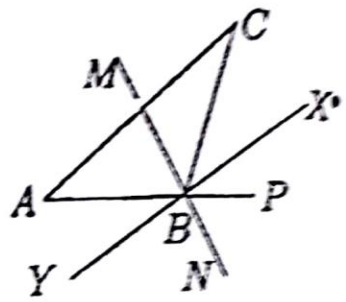
(39). In the adjoining figure ΔABC is right angled at B and $CD = CL$ and $AM = AD$. Calculate the measure of $\angle MDL$.



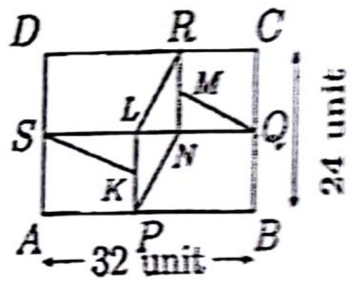
(40). $ABCD$ is a square. P, Q, R and S are some points on AB, BC, CD and DA. Prove that the perimeter of the quadrilateral $PQRS$ is less than the perimeter of the square.



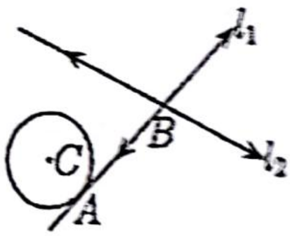
(41). ABC is a triangle. AB is extended to P. The line XY bisects $\angle CBP$. NM is perpendicular to XY at B. Show that the line MN bisects $\angle ABC$.



(42). $ABCD$ is a rectangle. In the figure, SLK, PLN, QNM, RNL are congruent right angled triangles. Find the area of each triangle.



(43). Given a circle, and two lines, draw all configurations for the relative positions of the circle and the lines. How many intersection points are determined by each configuration. What is the maximum points of intersections? One such configuration is shown here.



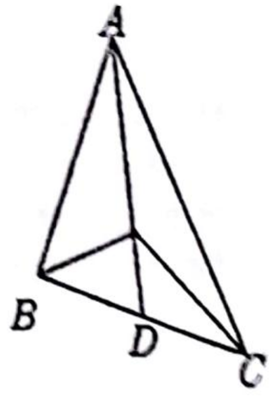
(44). In Question 43, if there are

(a) two circles and a line (b) two circles and two lines,

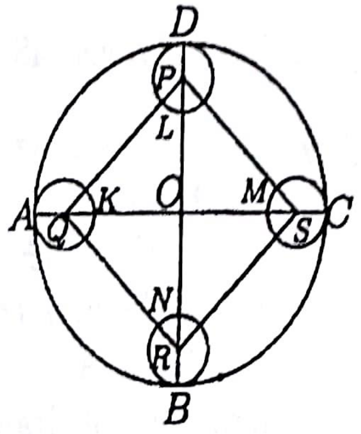
how many configurations can be got? What is the maximum number of points of intersection in each case?

(45). ABC is a triangle, D is the midpoint of BC, M is any point on AD. Show that area of ΔAMB = area of ΔAMC.

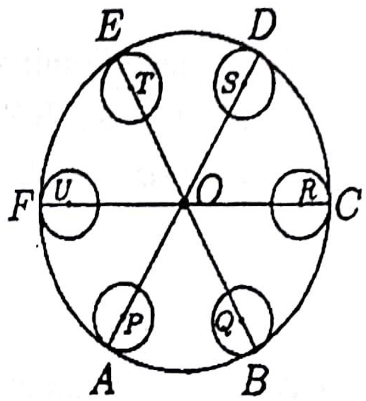
[Hint: There is just one perpendicular line (altitude) from a given point to a given line.]



(46). AC and BD are two perpendicular diameters of a circle with centre O. Four small circles are drawn with centres on AC and BD at P, Q, R and S respectively touching the circle at D, A, B and C respectively. The radii of the small circles are equal. Show that $PQRS$ is a square.



(47). In the adjoining figure, the three diameters AD, BE and CF make equal angles $\angle AOB$, $\angle BOC$, $\angle COD$, $\angle DOE$, $\angle EOF$ and $\angle FOA$ at O. The small circles touch the circle centre O at A, B, C, D, E and F as shown and they are of equal radii.



The centres of the smaller circles are $PQRSTU$.

(a) Show that $PQRSTU$ is a regular hexagon

(b) Show that $PUTO$ is a rhombus.

(c) Show that $OU = OP$

(d) Show that $PUTS$ is a trapezium and $PS = 2UT$

(e) Show that $PT = TR = PR$ and

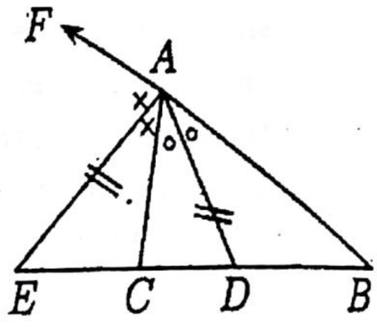
(f) Find the $\angle s$ of the triangle POR.

(48). If a configuration of 4 points $ABCD$ is given, then what is the maximum number of line segments, which are pairwise non intersecting except at A, B, C and D?

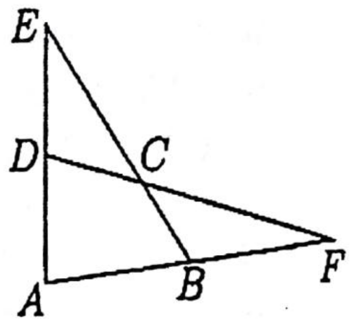
(49). Extend the number of points in Question (48) to (a) 5, (b) 6, (c) 7 and (d) n. Find the maximum number of non intersecting line segments which can be drawn using these points, which if they meet, meet at only the given points. (Hint: Make a table and generalise.)

(50). In Questions (48) and (49) count the total number of Δs formed.

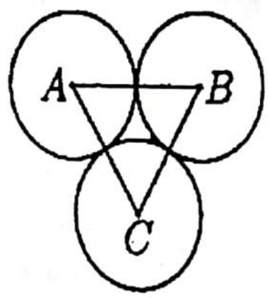
(51). In the adjoining figure, the bisector of $\angle CAB$ meets at CB at D, and the bisector of $\angle CAF$ meet BC extended at E. Calculate the value of $\angle ACB - \angle ABC$ if $AD = AE$. (Note that the figure is not drawn to scale).



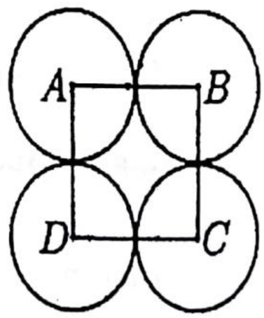
(52). In the adjoining figure, show that
$\angle DEC + \angle BFC =$
$\angle DCB - \angle DAB.$



(53). (a) A, B and C are the centres of three circles such that each circle touches the other two. If ABC is an equilateral Δ, show that the radii of the circles are equal.



(b) A, B, C and D are the centres of 4 touching circles as shown here.
If $ABCD$ is a square, show that the radii of the circles are equal.



Chapter 8

Miscellaneous

8.1 Combinatorics, Counting, Colouring and Measuring

(1). Write the numbers from 1 to 12 and group them in three groups so that each group has four numbers, and sum of the numbers in each group is the same. Show two such groupings.

(2). Numbers 1 to 9 are grouped into 3 groups, so that sum of the numbers in each group is the same and each group has 3 numbers. In how many ways can the grouping be done? Write all the possible groupings.

(3). A child is asked to write numbers from 1 to 5 using two colours, red and green. In how many ways can he write these numbers using the two colours?

(4). In how many ways 5 numbers can be written using three colours, using each colour at least once?

(5). A unit square is drawn and painted Green. Around this unit square, 3 more unit squares are annexed and painted Blue. This process of annexing unit squares and painting alternately Green and Blue is done in a 10×10 square. How many are painted blue? What is the number and colour of the unit squares annexed to 10×10 square to get 11×11 square?

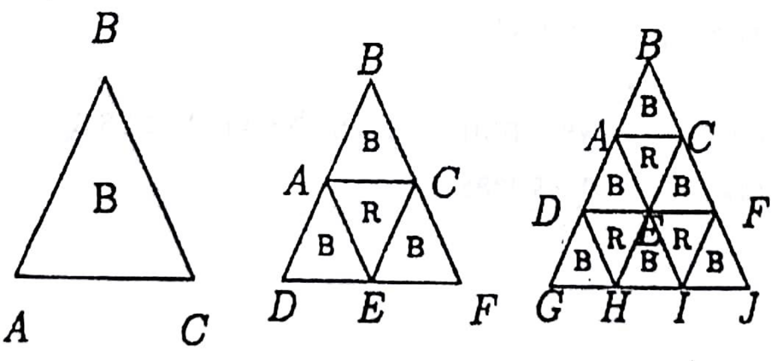
G	G	G
B	B	G
G	B	G

(6). In Question 5, if Green, Blue and Red colours are used alternately, give the corresponding answers to the questions asked.

(7). Starting with a unit square, bigger squares of sides 3, 5, 7 ... are obtained annexing 8, 16, 24, ... unit squares. Here we start painting the first unit square Green and the succeeding unit squares, which border the previous square are alternately painted with Red and Green. In this process, a 11×11 square is obtained. How many unit squares are painted green and how many are painted red? How many border unit squares are to be annexed to the 11×11 square to get a 13×13 square? What is the colour of the new unit squares in the 13×13 square? What is the colour of the border squares of a 101×101 square?

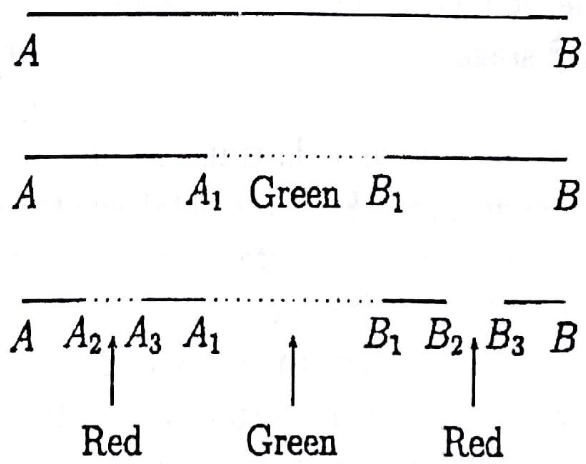
R	R	R
R	G	R
R	R	R

(8). Starting with an equilateral triangle of unit side, bigger equilateral triangles with side length 2 units, 3 units, ... are constructed. The first three stages are shown here:



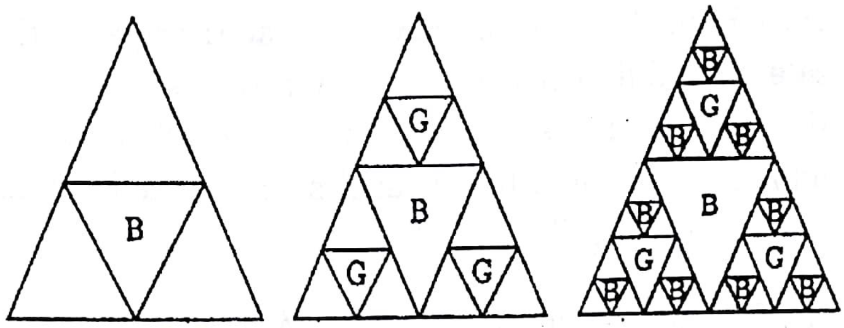
How many equilateral triangles will there be in the 10^{th} stage? To get the 11^{th} stage, how many unit triangles are to be annexed to the 10^{th} stage figure? These unit triangles are coloured starting with Blue, followed by Red, so that triangles with a common side have different colours. How many Blue unit triangles and Red unit triangles will there be in the 10^{th} stage? What is the number of red and blue triangles among the base unit triangles in the 10^{th} stage?

(9). AB is a line segment of 1 unit. In stage 1, it is divided into 3 equal parts and the middle part A_1B_1 is painted green. In each of stage 2, AA_1 and BB_1 is divided in to 3 equal parts, and the middle parts of AA_1 and BB_1 are coloured red. This process of dividing the segments and colouring is continued up to 6 stages. What is the total length of the green coloured segments and red coloured segments, if the segments are coloured alternately green and red? In the third stage, the 9 triangles obtained from the second stage are split and the 9 smaller triangles so obtained are coloured blue.



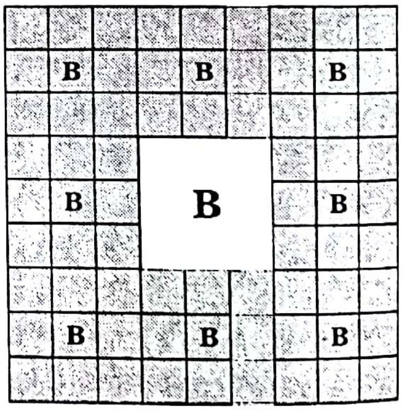
(10). Here are some figures: in the initial stage an equilateral triangle of side 1 unit is taken. In the first stage midpoints of sides are joined and another equilateral triangle is got and coloured Blue.

Repeat the first stage process to the three equilateral triangles formed at the corner and paint the three middle triangles Green.



If this process is continued upto 5 stages, using Blue and Green colours alternately, find the number of Green triangles, Blue triangles and uncoloured triangles.

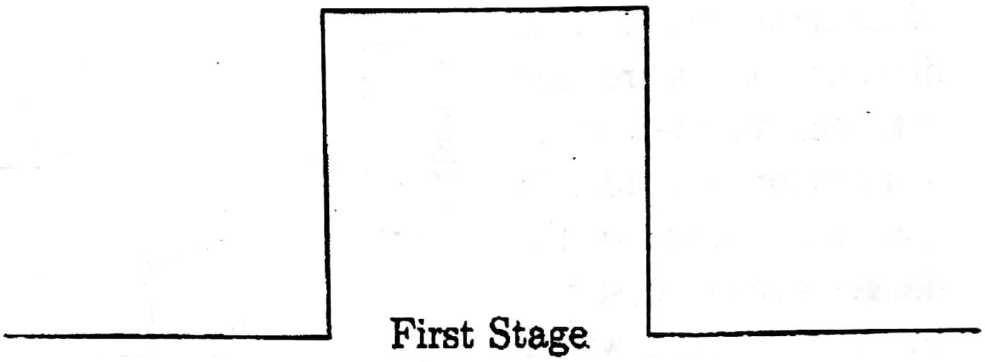
(11). Replace the triangle by a square, and get the middle square by drawing pairs of lines passing through points of trisection of the sides. Continue the process. The nine shaded squares are shown here. Find the side length of squares painted in the 5^{th} stage.



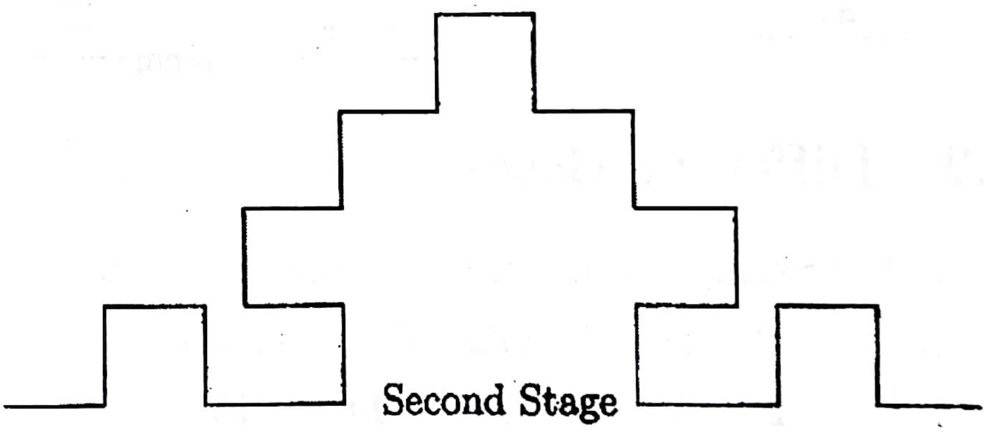
(12). Look at the sequence of diagrams given on the facing page. Find the total length of all the segments, in each stage from initial stage upto 3 stages, given the initial length to be 1 unit. Find the number of sides in each stage. Calculate the total length of all the segments in the 4^{th}, 5^{th} $\ldots 10^{\text{th}}$ stages. How many segments are there in the $10n^{\text{th}}$ stage? Can you generalise to the n^{th} stage?

Note: In the first stage, middle third of the portion gives a square above the initial line, with one side at the bottom deleted. This is repeated at every stage as shown

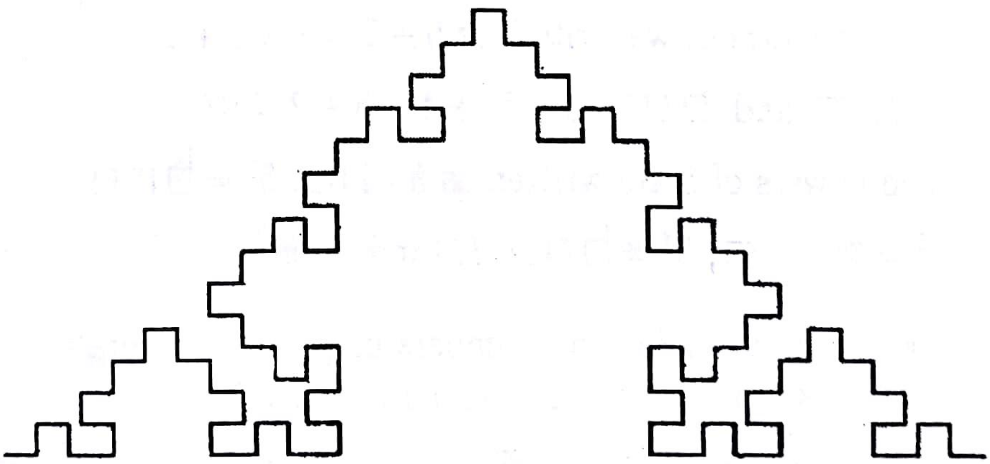
Initial Stage (0-stage)



First Stage

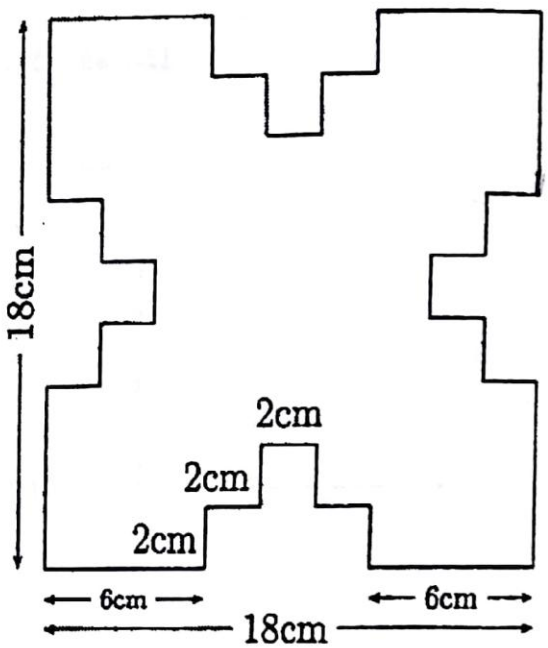


Second Stage



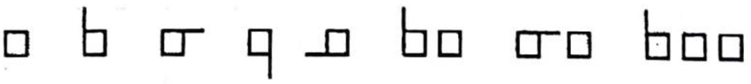
Third Stage

(13). An invitation has the following design. What is the perimeter of this design? What is the area? If three different colours are used with one for the interior and two for the border, in how many ways can the design be coloured, so that the adjacent sides with a common vertex are coloured differently.

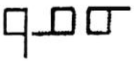


8.2 Different Bases

(14). In Fivi country, the numerals are written as follows:



0 1 2 3 4 5 10 25

In this method, we write 7 as $5+2 = 1\times 5+2 =$  and  $= 3\times 5^2 + 4\times 5 + 2 = 97$.

The powers of 5 are written as 5= , $5^2 =$ ,

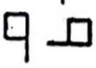
$5^3 =$ , $5^4 =$  and so on.

a) Write the following numbers using Fivi numerals.
(i) 8 (ii) 9 (iii) 12 (iv) 15 (v) 127

b) Change the following Fivi numerals into out decimal system

(i) 

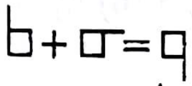
(ii) 

(iii) 

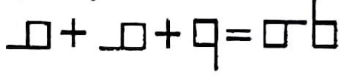
(iv) 
(v) 

c) Can you write all natural numbers using Fivi numerals?

In Fivi numerals the addition is done as follows:

(i) 

(ii) 

(iii) 

(Change the Fivi numerals into our system of numerals and verify the result.)

The above numerals are base 5 numerals. Here the base system is units, fives, five squares (or 25), five cubes (or 125) and so on. We require 5 different symbols as numerals for base 5 system. You can very well use 0, 1, 2, 3, 4 as the five numerals and write $5_{10} = 10_5$; $(13)_{10} = (23)_5$ and $(134)_5$ means $1 \times 5^2 + 3 \times 5 + 4 = (44)_{10}$. Similarly you can have base 6, base 7 ... base 12, base 16, ... systems. For base n, $n \geq 2$, we require n symbols. Here the base is represented as a subscript.

(15). Verify the following:

a) $24_{10} = 40_6$, $33_{10} = 53_6$

b) Convert the following numbers given in base 6 into base 10.
(i) 23_6 (ii) 223_6 (iii) 123_6 (iv) 321_6 (v) 54321_6

c) Change into base 6:
(i) 10_{10} (ii) 20_{10} (iii) 18_{10} (iv) 36_{10} (v) 38_{10}

d) Change into base 5:
(i) 54_6 (ii) 45_6 (iii) 32_6 (iv) 23_6 (v) 555_6

e) Change into base 6:
(i) 32_5 (ii) 23_5 (iii) 43_5 (iv) 34_5 (v) 123_5

(16). What is $5_6 \times 3_6$? If this product is converted into base 5, how should it be written?

(17). Find $5_6 \times (3_6 + 4_6)$ and $5_6 \times 3_6 + 5_6 \times 4_6$.

(18). Write the divisibility test (a) by 5 in base 5 (b) by 6 in base 6.

8.3 Alphamatics

In Alphamatics, digits are replaced by alphabets so that each alphabet represents a different digit. Solve the following alphamatics; (*i.e.*, find the values of the letters used in the problems, so that problem is correct.)

(19).
```
  BASE
 +BALL
 GAMES
```
Note: You have more than one solution. You can have the corresponding addition with or without carry over. Give at least one solution for each case.

(20).
```
 THREE
- FOUR
  FIVE
```
Here, it sounds ridiculous but in representing by numbers, it is a valid problem..

(21).
```
  GOD
+  IS
+NOW
 HERE
```
Note that this problem has no solution!
```
  GOD
+  IS
+  NO
WHERE
```

(22).
```
 ONE
+ONE
 TWO
```

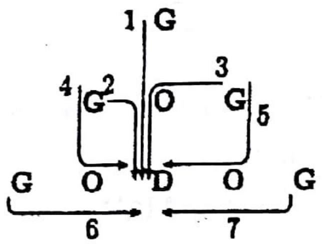
$$
\begin{array}{lr}
 & \text{FOOD} \\
(23). & +\ \text{FAT} \\
\hline
 & \text{DIETS} \\
\hline
\end{array}
$$

8.4 More Problems Using Letters

(24). In how many ways can the word MATHS written below as a pyramid be read?

Hint: The word GOD can be read in 7 ways.

```
    M
   MAM
  MATHM
 MATHTAM
MATHSHTAM
```



(25). If the word MATHEMATICS is written as above in how many ways can it be read? Can you generalise this for n letter words, where n is odd?

8.5 Matrices and coding

Matrices

A square array of numbers enclosed in parenthesis as shown below is called a square matrix (of numbers).

Examples:

(1) $\begin{pmatrix} 3 & 4 \\ 7 & 8 \end{pmatrix}$ (2) $\begin{pmatrix} 1 & 5 & 7 \\ 8 & 9 & 4 \\ 4 & 3 & 2 \end{pmatrix}$ are 2×2 and 3×3 square matrices.

They are called square matrices of 2^{nd} order and 3^{rd} order respectively. Addition of matrices is done by adding the corresponding entries of

two matrices of the same order.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

But multiplication is defined differently.

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}$$

Perhaps the arrows will help you to remember the definition of multiplication.

Note:. Matrix multiplication is not commutative *i.e.*, $A \times B \neq B \times A$, where A and B are two square matrices of the same order.
Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 5 & 6 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 1\times 5+2\times 0 & 1\times 6+2\times 7 \\ 3\times 5+4\times 0 & 3\times 5+4\times 7 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 20 \\ 15 & 43 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 6 \\ 0 & 7 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 21 & 28 \end{pmatrix}$$

(26). If $\begin{pmatrix} 2 & 3 \\ a & b \end{pmatrix} \times \begin{pmatrix} a & b \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 12 & 13 \\ c & d \end{pmatrix}$ find a, b, c and d.

(27). Given $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix}$ find a, b, c, d.

(28). If $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ find $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(29). If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ find $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

[Hint: Find the relation between a and d, and hence find such a matrix. One such matrix is $\begin{pmatrix} \frac{1}{4} & \frac{1}{16} \\ \frac{1}{8} & \frac{3}{4} \end{pmatrix}$. Verify.]

(30). If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and a, b, c, and d are non zero numbers, then find the condition for this equation to be true. One example:

$$\begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix} \times \begin{pmatrix} 6 & 9 \\ -4 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Find at least two more examples.

Coding

If you use numbers in the place of letters taking $A = 1,\ B = 2, \ldots Y = 25,\ Z = 26$, or $Z = 1, Y = 2 \ldots, B = 25,\ A = 26$ or start from any letter as 1 say $P = 1,\ Q = 2, \ldots Z = 11,\ A = 12, \ldots, O = 26$, and then convert your message in the form of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the message is now hidden in the matrices.

Now each of these matrices should be multiplied by another two by two matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $AD \neq BC$. Now you can read the secret message. The matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ given above is called an *Encoder*. The receiver should multiply by the Decoder which is

$$\begin{pmatrix} D/(AD-BC) & -B/(AD-BC) \\ -C/(AD-BC) & A/(AD-BC) \end{pmatrix}.$$

Also the order of conversion of the alphabets into numbers (say $A_1 - Z_{26}$, A_{26}. Z_1, $q(P_1, O_{26})$. The receiver then can read your message. Let us take the 4 lettered word GOOD. If you use $A_1 - -Z_{26}$, then the corresponding matrix is $\begin{pmatrix} G & O \\ O & D \end{pmatrix} = \begin{pmatrix} 7 & 15 \\ 15 & 4 \end{pmatrix}$ Using the encoder $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ we can get $\begin{pmatrix} 7 & 15 \\ 15 & 4 \end{pmatrix} \times \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 51 & 22 \\ 53 & 19 \end{pmatrix}$.

Now no body can guess what this is! To get the information back, the subscript tells how the message is converted. Multiply the message by the decoder of $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$ *i.e.*, $\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$.

$$\begin{pmatrix} 51 & 22 \\ 53 & 19 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 15 \\ 15 & 4 \end{pmatrix} = \begin{pmatrix} G & O \\ O & D \end{pmatrix}$$

$\Longrightarrow$ 'Good' is the message.

Caution: If the message matrix is multiplied by encoder *from the right*, one should multiply the coded matrix by the decoder from the *right only*.

(31). Decode the following message.

$\begin{pmatrix} 35 & 45 \\ 16 & 22 \end{pmatrix}$ $B_1 A_{26}$ dc $-$ 1, -2, 1 (Here you are given the decoder directly.)

(32). $\begin{pmatrix} 31 & 49 \\ 46 & 68 \end{pmatrix}$, $\begin{pmatrix} 38 & 56 \\ 24 & 39 \end{pmatrix}$ $Z_1 - A_{26}$ $dc2, -1, -1, 1$

Another method of coding

Write $A = 1, B = 2, \ldots Z = 26$. Here we use the sequence of prime numbers: 2, 3, 5, 7,.... We take the first letter of the message by the first prime number raise it to the power of that number which represents that alphabet or letter, and in general, n^{th} letter of the word by n^{th} prime number, raised to the power of its number representation.

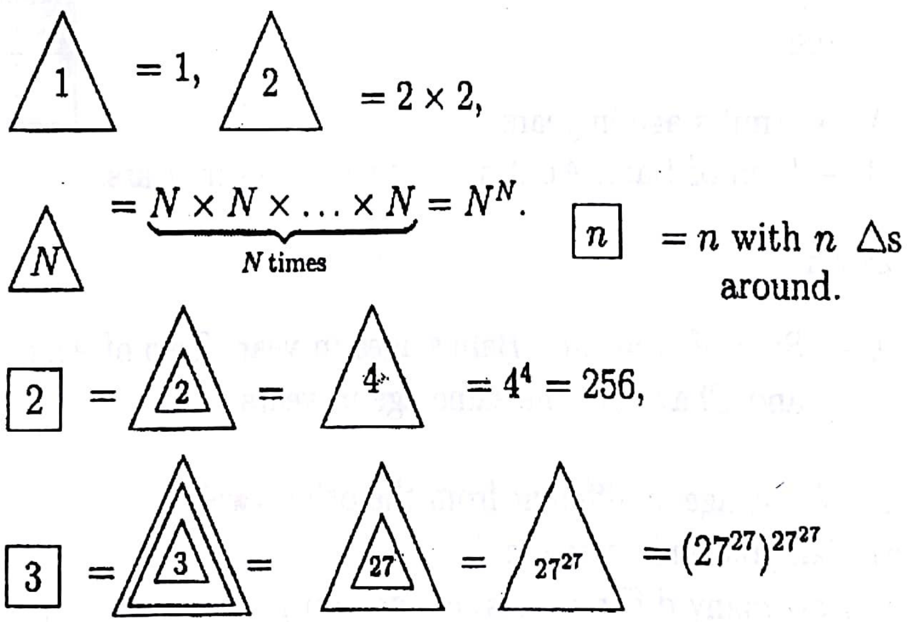
The word GOOD, as a coded message is $2^7\ 3^{15}\ 5^{15}\ 7^4$. The indices 7 15 15 4 give the message 'Good'. In practice, we should write the value $2^7.\ 3^{15}.\ 5^{15}.\ 7^4$ in standard form. The receiver first expresses the number in standard form as a product of powers of prime numbers and then decodes the message. Even for small messages the coded number becomes very large.

(33). Decode: $16875 \times 10^9 \times 390625$.

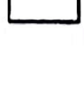
(34). Write yes we see (use the letters S V C for yes we see) in the above coding method.

Large Numbers

Read the following



(35). Find (a)  (b)  (c)

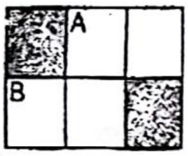
(36). Find out which is bigger:  or 

(37). Find out which is bigger: $\boxed{2} \times \boxed{2}$ or $\boxed{2} \times \triangle 3$

8.6 Some Puzzles

(38). A king had 3 wives. A pearl merchant visited the king and presented him 9 pearl necklaces, the cost ranging from 1 to 9 sovereigns (no fraction involved). Help the king distribute the necklaces to his three wives, so that each gets 3 necklaces and the total price of the necklaces received by each of his wives is the same.

(39). This is a crossnumber puzzle.



Across

A → Amit's age in years.
B → Sum of Balu, Amit and Chaya's ages in years.

Down

A → Sum of Amit and Balu's ages in year. Two of Amit, Balu and Chaya are the same age in years.

a) Whose age is different from the other two?
b) Can you find their ages?
c) How many different answers do you get?

(40). Look at the display of letters shown here:

A		D
B	G	E
C		F

a) Each letter represents a different digit taken from 0 to 9
b) The products ABC, BGE, DEF are equal.

What is the product of $A \times G \times F$?

8.7 Problems on Weights

(41). Find the minimum number of weights to be used to weigh from (a) 1 kg to 13 kg (b) 1 kg to 40 kg. Both sides of the pan can be used.

(42). There are 8 identical coins, one of which is lighter. Explain how the counterfeit coin (lighter coin) can be detected using a common balance, weighing exactly twice

(43). There is a counterfeit coin in a collection of 8 identical coins. It is not known whether the counterfeit coin is lighter or heavier. Explain how the counterfeit coin can be detected weighing exactly four times.

Part II

Solutions

CHAPTER 9

Problems on Number Properties

(1). The two digit number pairs that could be obtained using the digits 2, 4, 6 and 8 exactly once are

$$\begin{array}{ccl} & \nearrow & 68 \\ 24 & & \\ & \searrow & 86 \end{array} \qquad (24,68),\ (24,86)$$

$$\begin{array}{ccl} & \nearrow & 48 \\ 26 & & \\ & \searrow & 84 \end{array} \qquad (26,48),\ (26,84)$$

$$\begin{array}{ccl} & \nearrow & 46 \\ 28 & & \\ & \searrow & 64 \end{array} \qquad (28,46),\ (28,64)$$

$$\begin{array}{ccl} & \nearrow & 68 \\ 42 & & \\ & \searrow & 86 \end{array} \qquad (42,68),\ (42,86)$$

$$62 \begin{array}{l} \nearrow 48 \\ \\ \searrow 84 \end{array} \qquad (62, 48),\ (62, 84)$$

$$82 \begin{array}{l} \nearrow 46 \\ \\ \searrow 64 \end{array} \qquad (82, 46),\ (82, 64)$$

Thus we have 12 two digit number pairs (24, 68), (24, 86), (42, 68), (42, 86), (26, 48), (26, 84), (62, 48), (62, 84), (28, 46), (28, 64), (82, 46), (82, 64). The maximum sum is obtained when the two numbers have the maximum 10s digits, which are (62, 84), (82, 64). So maximum sum is 146. Similarly the minimum sum is found to be $26 + 48 = 28 + 46 = 74$. The maximum difference is $86 - 24 = 62$. The minimum difference is $62 - 48 = 14$.

(2). a) Hint: Take the four consecutive digits as a, $a+1$, $a+2$ and $a+3$, where $a+3 \leq 9$. Thus the maximum value of a can be 6.

The minimum value of a is 1. There are 6 such consecutive number groups (Find out what they are!). The maximum sum in this case in terms of a will be

$$\begin{array}{c|c} \text{Tens} & \text{units} \\ a+3 & a+1 \\ a+2 & a \\ \hline 2a+5 & 2a+1 \\ \hline \end{array}$$

which is $10(2a+5) + 2a + 1 = 22a + 51$, for each $a = 1$, 2, 3, 4, 5 or 6. The minimum difference is

$$\begin{array}{rcc} & \text{Tens} & \text{Units} \\ & a+2 & a \\ - & a+1 & a+3 \\ \hline & & \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{rcc} & \text{Tens} & \text{Units} \\ & a+1 & 10+a \\ - & a+1 & a+3 \\ \hline & & 7 \\ \hline \end{array}$$

Check that the maximum difference is 31.

b) For odd numbers take the 4 digits to be a, $a+2$, $a+4$ and $a+6$ with $a = 1$ or 3 and continue as in Question 2 a). The maximum sum is

Tens	units
$a+4$	$a+2$
$a+6$	a
$2a+10$	$2a+2$

i.e., $22a + 102$. If $a = 1$, the maximum sum is 124 and if $a = 3$ it is 168.

(3). If one three digit number is formed using three of the six digits, a second three digit number may be formed with the remaining digits. Thus, we first count the number of three digit numbers formed using any three of the six digits.

The three digit number pairs are

(123) (456)	(135) (246)
(124) (256)	(136) (245)
(125) (346)	(145) (236)
(126) (345)	(146) (235)
(134) (256)	(156) (234)

Thus there are 10 pairs of 3 digit numbers without repetition of digits.

But, for each combination of 3 digits, there are 6 numbers generated, giving rise to 36 number pairs for each of the above 10 three digit number pairs. For example,

$$(123) \rightarrow \quad (456)$$

$$(123) \rightarrow (465)$$
$$(123) \rightarrow (546)$$
$$(123) \rightarrow (564)$$
$$(123) \rightarrow (654)$$

and the digits (123) give rise to 6 numbers 123, 132, 213, 231, 312, 321. Thus for each of the numbers formed using the digits (123), there are six numbers, formed using the digits (456) and hence there are 36 pairs of numbers for the combination (123) and (456). Since there are 10 pairs giving rise to $10 \times 36 = 360$ pairs of 3 digit numbers, with out repetition of digits.

As before, to find the biggest sum, make the 100s place digits to be the maximum in both the numbers of the pair, then the next highest digits in 10s place and the other two digits left out in units place.

Thus the maximum sum is

	Huns.	Tens	Units			Huns.	Tens	Units
	6	4	2	$\Rightarrow$		5	4	1
+	5	3	1		+	6	3	2

(or any other choice, but retain 6 and 5 for 100s place, 4 and 3 for 10s place and 1 and 2 for units place.) which gives 1173. Similarly the minimum sum is

$$\begin{array}{r} 1\ 3\ 5 \\ +\ 2\ 4\ 6 \\ \hline 3\ 8\ 1 \\ \hline \end{array}$$

The maximum difference is

$$654 - 123 = 531$$

and the minimum difference is

$$412 - 365 = 47.$$

(Here pick up first 6, 5 as the 10s and units place of the number to be subtracted, *i.e.*, the biggest 10s and units place, then take the smallest digits 1, 2 for 10s and units places of number from which the subtraction is done, the numbers left are 6 and 5 differing by 1, and are placed as above.)

(4). Do it yourself.

(5). a) $(2-0)+(3-1)+(4-2)+\cdots+(101-99)$
$= 2+2+\cdots+2 \quad 100 \text{ times} = 200$
(*Note:* for any a, $a' - 'a = (a+1)-(a-1) = 2$)

b) $2+3+4\cdots+100+101$
$= (1+2+3+\cdots+100+101) - 1$
$= \frac{101\times102}{2} - 1 = 5150$

c) $0+1+2+\cdots+99 = \frac{99\times100}{2} = 4950.$

d) $2+4+6+\cdots+100$
$= 2(1+2+3+\cdots+50) = 50\times51 = 2550$

e) $2'+4'+6'+\cdots 100'$
$= 0'+2'+4'+\cdots+100'-0'$
$= 1+3+5+\cdots 101-1$
$1+3+5+\cdots 101$ is the sum of the first 51 odd numbers $= 51^2 = 2601$ and hence
$2'+4'+6'\cdots+100' = 2601-1 = 2600.$
Note: $2n-1$ is the n^{th} odd number.

$$2n-1 = 101 \Rightarrow n = 51.$$

and $1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = n^2$
i.e., the sum of the first n odd numbers $= n^2$.

f) $(1' - {}'2) + (3' - {}'4) + (5' - {}'6) + \cdots + (99' - {}'100)$
$= 1 + 1 + 1 \cdots + 1 = 99 \times 1 = 99$

g) $(1' + {}'1)' + {}'(2' + {}'2) + (3' + {}'3)' + \cdots + {}'(100' + {}'100)$
$= 3 + 3 + 7 + 7 + 11 + 11 + \cdots + 199 + 199$
$= 2(3 + 7 + 11 + \cdots + 195 + 199)$
$= 2(202 + 202 + + \cdots 202)$
$= 2 \times 25 \times 202 = 10100$

h) Follow the above method for subdivisions h) to l).

(6). $\left(a' + (a')'\right)' = (a + 1 + a + 2)' = 2a + 4$
${}'\left[(a')' + \left((a')'\right)'\right] = {}'[a + 2 + a + 3] = 2a + 4$
and hence $\left(a' + (a')'\right)' - \left[(a')' + \left((a')'\right)'\right] = 0$

(7). The value of
$99' + 100' + 101' = 100 + 101 + 102 = 303 = 3 \times 101 = 3 \times 100'$.
In general
$a' + (a')' + \left((a')'\right)' = 3a + 6 = 3(a + 2) = 3(a + 1)' = 3(a')'$

(8). $a + \left((a')'\right)' = a + a + 3 = (a + 1) + (a + 2) = a' + (a')'$

(9). $(a_1' - {}'a_1) + (a_2' - {}'a_2) + \cdots + (a_{100}' - {}'a_{100})$.
In general $a' - {}'a = (a + 1) - (a - 1) = 2$ for any a.
Thus the result is $2 + 2 + \cdots$ 100 times $= 200$.

(10). a)

$$25 = 20 + 5 = 5(4 + 1) = 5 \times 4'$$
$$\text{or} \quad 25 = 30 - 5 = 5(6 - 1) = 5 \times {}'6$$

b)

$$36 = 12 + 24 = 12(1+2) = 12 \times 2'$$
$$\text{or} \quad 36 = 12 \times 3 \ = 12 \times 2'$$

c)

$$27 = 81 - 54 = 9 \times (9-6) = 9 \times 3 = 9 \times {}'4$$
$$\text{or} \quad 27 = 81 - 54 = 27(3-2) = 27 \times 1 = 27 \times {}'2$$

etc. For prime numbers P, if $P = a + b$, there cannot be a common divisor for a, b greater than one, and hence $P = (a+b)$ cannot be factored with any factor > 1.

$$\therefore P = P \times 1 = P \times {}'2 \quad \text{or} \quad P \times 0'.$$

are the only two ways of expressing, using predecessor or successor.

(11). The digits 2, 4, 6 and 8 give the following two digit number pairs:

(24, 68)	(42, 86)	(24, 86)	(42, 68)
(26, 48)	(26, 84)	(62, 48)	(62, 84)
(28, 46)	(28, 64)	(82, 46)	(82, 64)

(*Note:* each number in the number pair should have just two of the 4 given digits and no other digits should be used.)
In the first row of pairs of numbers, 42×86 is the greatest. In the 2nd row, 62×84 is the greatest and in the 3rd row, 82×64 is the greatest. So we should find which of the products, (a) 42×86 (b) 82×64 (c) 62×84 is the greatest. Comparing (a) and (b)

$$42 \times 86 = 42 \times 82 + 42 \times 4$$
$$82 \times 64 = 82 \times 42 + 82 \times 22$$

and hence clearly 82×64 is greater than 42×86.

Comparing 82×64 with 62×84.

$$\begin{aligned} 82 \times 64 = (84-2)64 &= 84 \times 64 - 2 \times 64. \\ &= 84 \times 62 + 84 \times 2 - 2 \times 64. \\ &= 84 \times 62 + 20 \times 2 > 84 \times 62. \end{aligned}$$

Thus 84×64 gives the maximum value for the product.
Note: If (a, b) and (c, d) are any two pairs of numbers with $a + b = c + d$, and $a - b < c - d$, then the product $ab > cd$.
In the above problem, considering the pairs $(82, 64)$ and $(62, 84)$. $82 + 64 = 62 + 84 = 146$ and $82 - 64 = 18$ and $84 - 62 = 22$ and we found $82 \times 64 > 84 \times 62$.
If $a - b = c - d$ and $a + b > c + d$ then $ab > cd$.

(12). (a) As in problem 3, there are 20, 3 digit combinations and hence 10 pairs of 3 digit combinations using each digit 4, 5, 6, 7, 8, 9 exactly once. As each 3 digit number in each pair can be permuted in 6 ways, a total of 360 pairs are available.

Note: If a, b, c, d are natural numbers $a > b$, $c > d$ and $a + b = c + d$ and $a - b < c - d$ then $ab > cd$.
(b) Consider the pair (987, 456) with sum 1443. By interchanging (9, 4), (8, 5), (7, 6), we get other pairs with the same sum including the pair (456, 987). Among these the maximum product is where the difference is minimum *i.e.*, (956, 487).

(13). (a) The perimeter of a rectangle is $2l + 2b = 36$ or $l + b = 18$, where l, b are length and breadth.

S.No	l units	b units	A Sq.units
1	17	1	17
2	16	2	32
3	15	3	45
4	14	4	56
5	13	5	65
6	12	6	72
7	11	7	77
8	10	8	80
9	9	9	81

Thus there can be 9 different rectangles with 1 rectangle having both sides of equal length ie, it is a square and in this case where $l - b = 0$, the product $l \times b = 81$ is the maximum (refer *Note:* in problem No 12). Do the other subdivisions.

(14). (c) Area of the rectangle with integer side is 196.

l units	b units	Perimeter $2(l+b)$ units
1	196	$2 \times 197 = 394$
2	98	$2 \times 100 = 200$
4	49	$2 \times 53 = 106$
7	28	$2 \times 35 = 70$
14	14	$2 \times 28 = 58$

From the above table we see that when the area of the rectangle is fixed, the perimeter is minimum when the sides are equal (*i.e.*, it is a square).
Note: If $a \times b$ is fixed, then $a + b$ is minimum when $a = b$. Do the subdivisions (a) and (b).

(15). (a) Hundreds place of the first number for addition problem can be formed using any one of 5 digits thus it has 5 choices. Similarly for tens place $(5-1)=4$ and for unit place $(5-2)=3$ choices as repetition is not allowed. Thus the first 3 digit number (any 3 digit number) can be formed using the given five digits in $5 \times 4 \times 3 = 60$ ways.

The addend 2 digit number can be got in 2 ways. Thus the total number of addition problem will be $60 \times 2 = 120$. To get the maximum sum in addition problems, use the biggest digit for 100s place followed by tens places. Thus the biggest sum in this case will be

$$\begin{array}{r} 652 \\ +\ \ 43 \\ \hline 695 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{r} 642 \\ +\ \ 53 \\ \hline 695 \\ \hline \end{array} \quad \begin{array}{r} 653 \\ +\ \ 42 \\ \hline 695 \\ \hline \end{array}$$

(b) For subtraction problems also since the first number is a 3 digit number the same number of problems *i.e.*, 120 problems can be constructed. To get the least difference use the least values of digits for hundreds, tens and unit places of the first number and biggest value for tens and unit places for the 2nd number. *i.e.*,

$$\begin{array}{r} 234 \\ -\ \ 65 \\ \hline 169 \\ \hline \end{array}$$

(16). Do it yourself.

(17). When 9 digit numbers are added, the maximum carry over occurs when all the numbers are 9. Thus the maximum carry over is 8. Minimum carry over is 0 (Why?) Do the other subdivisions.

(18). Let a_1, a_2, a_3 and a_4 are the four single digit numbers. Let us find the different carry over when these numbers are added.

a_1	a_2	a_3	a_4	Sum	Carry over
9	9	9	9	36	3
9	9	9	8	35	3
9	9	8	8	34	3
9	8	8	8	33	3
8	8	8	8	32	3
9	9	9	7	34	3
9	9	8	7	33	3

Complete the table and find for what different value of a_1, a_2, a_3 and a_4 you get this carry over. If all the 4 numbers a_1, a_2, a_3 and a_4 are different we get $9+8+7+6 = 30$ and the carry over is 3 and no other combination with *distinct* a_1, a_2, a_3 and a_4 can give the carry over 3. (Why?)

(19). For you to do.

(20). For you to do.

(21). If nine 9s are taken, the sum is 81. To get a carry over 8, this sum might go up to 89 (ten 9s give 90). So the maximum nines that could be taken is 9 and any other non zero digit along with carry 8 gives nine 9s.

For a carry of 5, six 9s give 54, and this sum could go up to 59 by adding four more non zero single digit numbers. So the sum of the non zero digits must be less than or equal to 5, and the maximum nines that could be used is 6 and the other four non zero numbers are 1, 1, 1, 2.

(22). (a) Do it yourself

(b) If all the 9 numbers are taken to be 5 (> 4), we get $9 \times 5 = 45 < 52$ and if all the 9 numbers are 6(> 4) gives $9 \times 6 =$

$54 > 52$. You can see that the maximum 5s we can use is 6 (Why?)

$$52 - 6 \times 5 = 22 = 8 + 8 + 6.$$
$$\begin{aligned}\therefore 52 &= 5+5+5+5+5+5+6+8+8\\ &= 5+5+5+5+5+6+6+7+8\\ &= 5+5+5+5+6+6+6+7+7\end{aligned}$$

Find the other solutions.

(23). (a) $9 \times 9 = 81$ and 8 is the maximum carry over; $1 \times 1 = 1 = 01$ and the minimum carry over is 0. By suitable combination of 2 single digit numbers, the product may yield the other carry overs also.

(b) When four single digit numbers are multiplied. When repetition allowed, it gives the maximum product $9 \times 9 \times 9 \times 9 = 6561$, so that the digits immediately to the left of units place are 6, 5, 6 and 656 *is the maximum value.* However if repetition is not allowed you can try 1, 2, 3, 4, ..., 1, 2, 3, 5, ..., 1, 3, 4, 5, ..., 1, 7, 8, 9, ..., 2, 3, 4, 5, ..., and find the digits to the left of units place.

The maximum value in this case is $6 \times 7 \times 8 \times 9 = 3024$, and the digits to the left of unit digit 4 is 302.

The minimum is $1 \times 2 \times 3 \times 4 = 24$ and the single digit 2 lies to the left of units place.

(24). For the 3 digit product, we get 111, 112, ..., 118, 119 ...222, ..., 998, 999. $111 = 3 \times 37$, 37 being a prime cannot be expressed as a product taking 112, we have $1 \times 2 \times 56 = 1 \times 2 \times 7 \times 6$. This leads to $224 = 1 \times 4 \times 7 \times 6$ and so on.

We shall show here for at least one 4 digit numbers, the above property. $1008 = 2 \times 8 \times 9 \times 7$. Find at least 2 more four digit

numbers having this property. ⋆ This problem will develop factorizing a number, and looking at the number you can find if there is a two or 3 digit prime number(s) as a factor(s).

(25). Do using the definition

(26). First we shall discuss the problem with digital sum 12 generally, $12 = \underbrace{1+1+1+\ldots+1}_{12 \text{ numbers}}$ or 10 ones and one, two, ... 2 ones and 2 fives ... Again by inserting zeros between the digits of the numbers we can have an infinitely many numbers to have a digital sum 12. The smallest number with digital sum 12 is 39; If the digits lie between 2 and 8, we have $336, 363, \ldots$ have digital sum 12. Here we shall form a tabular column.

Digits used	Numbers	
3 3 3 3	3333	(1)
3 3 6	336 363 633	(3)
3 4 5	345 354 453 435 534 543	(6)
4 4 4	444	(1)

There are thus $1+3+6+1 = 11$ numbers and first 1 number is a 4 digit number and the rest are 3 digit and the number of even numbers are 4 is 336, 354, 534, and 444 and these 4 numbers are divisible by 6 and the other 7 are divisible by 3 none of the numbers is divisible by 9. Two are divisible by 4 (336 & 444). Just one number 336 is divisible by 8 also by 24.

Sum of these 11 numbers is

$$\begin{array}{r}
3\;3\;3\;3\\
4\;4\;4\\
3\;3\;6\\
3\;6\;3\\
6\;3\;3\\
3\;4\;5\\
3\;5\;4\\
5\;3\;4\\
5\;4\;3\\
4\;3\;5\\
4\;5\;3\\
\hline
7\;7\;7\;3\\
\hline
\end{array}$$

and the digital sum of 7773 is 24. Digital root is 6. The sum of the digital sum is 132. The digital root of the sum of the digital sum is 6. The sum of the digital roots is 33 and the digital root of the sum of the digital root is also 6. The only numbers, the digits of which lie between 2 & 8 and without repetition, to give a digital sum are 345, 354, 435, 453, 534, 543 and all these 6 numbers are divisible by 3,
None is divisible by 4.
~~2~~ are divisible by 6.
2 are divisible by 5 (also by 15).
sum of the numbers is 2164; The digital sum of the sum of these 6 numbers is 18 and the digital root of this sum is 9. The sum of the digital sums of these numbers is 72. Again the digital root of this sum is also 9. *Note:* In all these 11 numbers, there are 16 threes, 9 fours, 6 fives, 3 sixes which total up to $16 \times 3 + 9 \times 4 + 6 \times 5 + 3 \times 6 = 132$.

(27). (a) A 11 digit number with all its digits 9 has the digital sum 99 and this number has the minimum number of digits to yield a digital sum 99.

(e) The number with digital sum 10,000 and with minimum number of digits:

$$10000 \div 9 = Q\ 1111 \quad R\ 1$$

$\therefore$ $10000 \div 9$ gives quotient 1111 and remainder 1.

A number with 1, 111 digits all of which are 9s give the digital sum 9999, to get the digital sum 10,000, we should have one more digit, 1. Thus there are at least 1112 digits in the number whose digital sum is 10,000.

Note: you can have 1110 nines and, one 8 and one 2 in the 1112 digits also give the digital sum 10,000 Verify.

You can have still a number with more than *1112* digits to give a digital sum 10,000. But, the *minimum number of digits should be 1112.* You can have a pattern of no of digits starting with digital sum 10, 100, 1000, 10,000, ...

Digital sum	Minimum number of digits in the number
10	2 digit number
100	12 digit number
1000	112 digit number
⋮	⋮

Can you find such patterns for digital sums 9, 99, 999, ...?

(28). (c) 5 digit numbers with digital sum 4.

$4 = 1 + 1 + 1 + 1, = 1 + 1 + 2, = 1 + 3 = 4, 2 + 2 = 4$

Thus we can use 4 ones and a zero, 2 ones, 1 two and 2 zeros, one 1, one 3 and 3 zeros, one 4 and 4 zeros and two 2s and three zeros. Thus the answer is

(1) 11110, (2) 11101, (3) 11011, (4) 10111.
(5) 11200, (6) 11020, (7) 11002, (8) 10120,
(9) 10102, (10) 10210, (11) 10201, (12) 10012,
(13) 10021, (14) 12100, (15) 12010, (16) 12001,
(17) 21100, (18) 21010, (19) 21001, (20) 20101,
(21) 20110, (22) 20011, (23) 13000, (24) 10300,
(25) 10030, (26) 10003, (27) 31000, (28) 30100,
(29) 30010, (30) 30001, (31) 22000, (32) 20200,
(33) 20020, (34) 20002, (35) 40000.

Thus there are 35 five digit numbers whose digital sum is 4. Do the other subdivisions.

(29). Let the number of the digits 4 appearing in the 1000 digits numbers be a. To get the maximum number of 4s used, we can have the other non zero digits to be 1 say there are b 1s used. Thus we have

$$a + b = 1000. \tag{9.1}$$

$$4a + b = 1201 \quad \text{or} \quad 1202 \ldots 1209. \tag{9.2}$$

Eq (9.1) and (9.2) $\Rightarrow$

$$3a = 201, \quad \text{or } 202 \ldots 209.$$

$$\Rightarrow a = 67, \quad \text{or } 68, \text{ or } 69, \ldots$$

Taking the digital sum as 1207; if 69 (that is the maximum number) 4s are used and $1000 - 69 = 931$ ones are used. We get the digital sum as

$$69 \times 4 + 931$$

$$276 + 931 = 1207$$

Thus the maximum number of 4s used is 69. All the other digits may be 1 (or 69, 4s, 930, 1s and one 3 gives 276+930 +3=1209 or 69, 4s, 929, 1s, and 2 2s etc.)

(30). Using two 2s and one five we get (1) 225 (2) 252 (3) 522 and using two 5s and one two we get (4) 552 (5) 525 (6) 255. Thus there are six 3 digit numbers with at least one five and one two. Their sum is 2331 and

$$2331 = 3 \times 777 = 9 \times 7 \times 37$$
$$= 3^2 \times 7 \times 37.$$

The divisors of this number is 1, 3, $3^2 = 9$, 7, 37, $\boxed{21}$, 63, 111, 333, 259, 777, 2331.
Note: $21 = 2 + 2 + 5 + 5 + 5 + 2$.

(31). Use the same method as in Problem 30.

(32). Use the same method as in Problem 30.

(33). Try yourself.

(34). Try yourself.

(35). Try yourself.

(36). Try yourself.

(37). By Euclidean algorithm, we get
Dividend = divisor × Quotient + remainder
⇒ divisor × Quotient = Dividend − remainder.
In this Question, divisor is a 2 digit number and Quotient a 3 digit number with unit digit 9. Thus,

$$\underbrace{\square\square}_{\text{Divisor}} \times \underbrace{\square\square}_{\text{Final quotient}}9 = 50448 - 57 = 50391$$

$$\text{Again } \underbrace{\square\square}_{\text{Divisor}} \times \underbrace{\square}_{\text{Partial quotient}} = 504 - 9 = 495$$

Thus one of the 2 digit divisors of 495 will be the divisor $45 \times 11 =$ 495, $= 99 \times 5 = 9 \times 55$, $495 = 3 \times 165$, $= 9 \times 55$, $= 11 \times 45$.

Since Divisor $\times$ Final quotient ends in 1, we cannot have the divisor end in 5, so the divisor should be 99, so that single digit partial quotient is 5.
Now $50391 \div 99$ gives 509.
Thus the divisor is 99 and Quotient is 509. Follow the same method to solve (b) sub division.

(38). a, b, c are consecutive digits.

$$\begin{array}{cccc} & a & b & c \\ & c & b & a \\ & \square & \square & \square \\ \hline 1 & 2 & 4 & 2 \\ \hline \end{array}$$

Let us take them as x $x+1$ $x+2$ and the boxes can be taken as k, l, m so that the sum will look like

$$\begin{array}{cccc} & x & x+1 & x+2 \\ & x+2 & x+1 & x \\ & \square & \square & \square \\ & k & l & m \\ \hline 1 & 2 & 4 & 2 \\ \hline \end{array}$$

Here x and k, l, m are single digit numbers. So

$$x+2+x+m = 2x+2+m = 12 \quad \text{or} \quad 22$$
$$\Rightarrow 2x+m = 10 \quad \text{or} \quad 20$$

Let us first take $2x+m = 10$. m should be even, so m is 2, 4, 6 or 8, if $m = 2$, then $x = 4$ and the consecutive numbers are 4, 5, 6 and $m = 2$, is not one of them. Similarly, you can check that

m can't be 6 or 8. So taking $m = 4$, we get $x = 3$.
The consecutive numbers being 3, 4, 5 and $m = 4$ is one of them.
Now the sum becomes

$$\begin{array}{rcccc} \text{Carry} & & \boxed{1} & \boxed{1} & \\ & & 3 & 4 & 5 \\ & & 5 & 4 & 3 \\ & & 3 & 5 & 4 \\ \hline & 1 & 2 & 4 & 2 \\ \hline \end{array}$$

Thus the value of the last boxes are 3, 5, 4. The problem is fully solved in this case.
For $2x + m = 20$, show that no single digit value for n & m satisfy the given problem. [If $m = 8$, $x = 6$, the problem looks like

$$\begin{array}{cccc} & 6 & 7 & 8 \\ & 8 & 7 & 6 \\ & \boxed{k} & \boxed{l} & 8 \\ \hline 1 & 2 & 4 & 2 \\ \hline \end{array}$$

and we are not getting k and l to be either of 6&7. Again try with $m = 6, 4, 2$ and show that no value of m satisfy the given problem.]

(39). Do as in problem 38.

(40). Do as in problem 38.

(41). The highest power of 2:

$$\begin{aligned} N &= 36 \times 48 \times 26 \times 23 \times 18 \times 225 \\ &= 2^2 \times 3^2 \times 2^4 \times 3^1 \times 2^1 \times 13^1 \\ &\quad \times 23^1 \times 2^1 \times 3^2 \times 3^2 \times 5^2 \end{aligned}$$

$$= 2^8 \times 3^7 \times 5^2 \times 13^1 \times 23^1$$

2^8 is a divisor of $2^8 \times 3^7 \times 5^2 \times 13^1 \times 23^1$ but not 2^9. Thus highest power of 2 dividing the number is 8.
The highest power 3 dividing the number is 7 *i.e.*, 3^7 divides N, but $3^8 \times N$. The highest power of $6 = 2 \times 3$, dividing N is 7 $(2^7 \times 3^7 = 6^7)$

(42).

$$N_1 = 2^8 \times 5^9 = 10^8 \times 5$$
$$N_2 = 2^{16} \times 5^{13} = 10^{13} \times 2^3$$
$$N_3 = 4^{14} \times 5^{28} = 2^{28} \times 5^{28} = 10^{28}$$

Thus 10^8 divides $2^8 \times 5^9$, but not 10^9. So 8 is the highest power of 10, dividing N, and N_1, written in standard form has 8 zeros at right end of N_1. Similarly 10^{23} divides N_2, but 10^{14} does not divide N_2. There are 13 zeros at the end of N_2 and $N_3 = 2^{28} \times 5^{28} = 10^{28} = 1$ followed by 28 zeros.
Note: The number of zeros at the end of a number is the highest power 10, dividing the number.

(43). $10! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 = 2^8 \times 5^2 \times 3^4 \times 7^1$ is the prime factorisation of 10! and the highest power of 2 dividing 10! is 2^8 and the highest power 10, dividing 10! is 2.

To find the highest power of 10 dividing 100! We should find the highest power 5 dividing 100! Clearly the highest power of 2 dividing 100! is more than the highest power of 5, dividing 100! There are $100 \div 5 = 20$ fives, and these 20 fives have 4, 5^2. Thus 100! is divisible by $_{10}(20 + 4) = {}_{10}24$. *i.e.*, There are 24 zeros at the end of 100!

[Every fifth number, gives a power of 5, 5, 10, 15, 20, 25, 30, 35, $\ldots 50 = 2 \times 25 = 2 \times 5^2 \ldots 75 = 3 \times 5^2 \ldots 95, 100 = 5^2 \times 4$]

You can now find every 5^{th} number of first n multiples of 5, is divisible by 25, and every 5^{th} number in the first n multiple of 25 and so on. Thus we can find the highest power as

$$\left[\frac{100}{5}\right]+\left[\frac{100}{25}\right]+\left[\frac{100}{125}\right]+\cdots$$

$\left[\frac{100}{5}\right]$ is the biggest integer less than or equal to $\frac{100}{5}$, here it is 20, $\left[\frac{100}{25}\right]=4$, $\left[\frac{100}{125}\right]=0$
Therefore the highest power power of 5 dividing 100! is 24

[Example: Highest power of 5, dividing 112! is

$$\begin{aligned}&\left[\frac{112}{5}\right]+\left[\frac{112}{25}\right]+\left[\frac{112}{125}\right]+\cdots\\&=22+4+0+0+\cdots\\&=26\end{aligned}$$

$$\left[\frac{112}{5}\right]=22,\left[\frac{22}{5}\right]=4,\left[\frac{4}{5}\right]=0$$

and the highest power of 10, dividing 112! is 22+4= 26]

(44). Unit digit of 10! as well as 100! is 0

$$\begin{aligned}10s\ \text{digit of}\ 10! &= 10^2\times 2^5\times 3^4\times 7^1\\&=10^2\times 32\times 81\times 7\end{aligned}$$

is the number $(32\times 81\times 7)$ followed by 2 zeros. Thus tens place is also zero. For 100! has 24 zeros at the end last two digits are clearly 0.
$7! = 10^1\times 2^3\times 3^2\times 7^1 = 10^1\times 504 = 5040$. The unit digit is 0 and the 10s place is 4.
Note: $N!$ if $5\le n\le 90$. Then the unit place of $N!$ alone is zero. That is there is just one zero at the end of the number. To

find the 10s place, multiply the factors other than 10, together getting $N = 10^1 \times n$ and the unit digit of $n = 2^k \times 3^l \times 7^n$, forms the 10s place of N.

(45). $3! \times 5! \times 7! = n!$

$$\underbrace{1 \times 2 \times 3}_{3!} \times \underbrace{1 \times 2 \times 3 \times 4 \times 5}_{5!} \times \underbrace{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}_{7!}$$

$$= 7! \times 8 \times 15 \times 2 \times 3$$
$$= 7! \times 1 \times 2 \times 3 \times 4 \times 5 \times 6$$
$$= 7! \times 8 \times 9 \times 10$$
$$= 10!$$

(46). $\dfrac{1^2 \times 2^2 \times 3^2 \times \cdots \times 98^2 \times 99^2}{(100!)^2}$

$$= \frac{(1 \times 2 \times 3 \times \cdots \times 98 \times 99) \times (1 \times 2 \times 3 \times \cdots 98 \times 99)}{(100!)^2}$$

$$= \frac{99! \times 99!}{100! \times 100!} = \frac{1}{100 \times 100} = \frac{1}{10{,}000}$$

(47). L.H.S. $= \dfrac{n!}{(n-1)!} + \dfrac{(n-1)!}{(n-2)!} + \dfrac{(n-2)!}{(n-3)!} + \cdots + \dfrac{1!}{0!}$

$$= n + (n-1) + (n-2) + \cdots + 1$$
$$= 1 + 2 + 3 + \cdots + (n-1) + n$$
$$= \frac{n(n+1)}{2} = \frac{n \times n! \times (n+1)}{2! \times n!}$$
$$= \frac{n \times (n+1)!}{2! \times n \times (n-1)!} = \frac{(n+1)!}{2!(n-1)!} = \text{R.H.S}$$

(48). Clearly ten thousandth digit cannot be zero, and if 4 is placed in unit place, it will mean that there are 4 fours. Thus 3 other

digits should get 4. If thousandth place has 4, which means that it will show that there are four ones. But there are only 5 digits so, 4 can't be in units.

A little thinking will guide you to place zeros in t & u. That is there are no 3s or 4s in the number. Since there are 2 zeros one in units place the other in tens place. 10 thousandth places takes the digit 2

TTh	Th	H	T	U
2		:	0	0

Now we should find what should be H. Since there is 2 in TTh place, write 1 (one two) hundreds place, but then, TTh place, represents number of 1s. But if we write '1' there are 2 1s in the number. So 2 should be placed, but now it is a contradiction.

So H can have another 2, giving total number of 2s in the given number becomes 2, and new Th takes the value '1'. The number of ones is 1, that it self is in Th places.

So the number is

TTh	Th	H	T	u
2	1	2	0	0

Can you have any other answer? Try!

(49).

TTh	H	T	u	
1	8	6	9	×...□ □ □
1	9	4	7	

Since unit digit of the product is 7 unit digit of the divisor should be 3 (as $3 \times 9 = 27$) 7 is in the units place and 2 is carried over to 10s place.

Now do the 1st step in multiplication.

$$\begin{array}{r}
1\;8\;6\;9 \times \square\;\square\;3 \\
\hline
5\;6\;0\;7 \\
\\
\hline
4\;7
\end{array}$$

To get '4' in 10s place, we should multiply 9, by 10s place divisor, so that product end in 4 [since the tens place of partial product is 0]

Since 9×6=54, so the second or 10s place of the divisor is taken as 6.

$$\begin{array}{r}
1\;8\;6\;9 \times \square\;\square\;6\;3 \\
\hline
5\;6\;0\;7 \\
1\;1\;2\;1\;4 \\
\\
\hline
9\;4\;7
\end{array}$$

The hundreds place should be 9 we already have $6 + 1 = 7$, so the hundreds place of the partial answer should be $9 - 7 = 2$, 9×8=72. Thus the 100s place of the divisor is 8.

$$\begin{array}{r}
1\;8\;6\;9 \times \square\;8\;6\;3 \\
\hline
5\;6\;0\;7 \\
1\;1\;2\;1\;4\;0 \\
1\;4\;9\;5\;2\;0\;0 \\
\\
\hline
9\;4\;7
\end{array}$$

Now the thousands place addition yields 12, and since we should get 1 in the thousands place, we should add 9 to 12 *i.e.*, 12+9=21

then 1 will be in thousand place. To get 1. we should multiply '9' of 1869 by 1 which is the thousands place of the divisor.

Now the problem is completed.

			1	8	6	9	×	1	8	6	3
			5	6	0	7					
	1	1	2	1	4	0					
1	4	9	5	2	0	0					
1	8	6	9	0	0	0					
3	4	8	1	9	4	7					

So the smallest number by which 1869 is to be multiplied to get the product ending in 1947 is 1863 you can take any number ending in 1863, and multiplying 1869 by this number. The least 4 digits of the product will be 1947.

(50). Solve yourself as explained in Problem 49.

(51). Solve the problem following the examples given.

(52). Solve the problem following the examples given.

(53). The answer is grandfather's year of birth 1936. Father's year of birth 1964. Note $1936 = 44^2$, $36 = 6^2$, $64 = 8^2$. Father's age in 2000 is 36. Grandfather's age in 2000 is 64.

(54). Observe the example and solve.

(55). Observe the example and solve.

(56). Adding even numbers, we always get an even number and hence only even number can be obtained, but not odd numbers.

(57). The numbers 6 and 9 are not relatively prime. Their gcd is 3, and you can get all multiples of 3 example $42 = 9+9+9+9+6$.

(There are many ways of writing 42 using 9 and 6 and addition and/or subtraction.)

(58). Since gcd of 25 and 15 is 5, All multiples of 5 can written.
Note: if a and b are any two numbers relatively prime to each other, you can write all natural numbers (or integers) using these two numbers a and b.

(59). (a) Standard form of the sum of the given number is
$10(a_1 + a_2 + a_3) + (b_1 + b_2 + b_3) = 200.$

Given Nos		Digits replaced by 10 Complement	
T	u	T	u
a_1 a_2 a_3	b_1 b_2 b_3	$(10 - a_1)$ $(10 - a_2)$ $(10 - a_3)$	$(10 - b_1)$ $(10 - b_2)$ $(10 - b_3)$
$(a_1 + a_2 + a_3)$	$(b_1 + b_2 + b_3)$	$[30 - (a_1 + a_2 + a_3)]$	$[30 - (b_1 + b_2 + b_3)]$

Standard form of the sum of the 10 Complement digits numbers is

$$\begin{aligned}
&= 10[30 - (a_1 + a_2 + a_3)] + [30 - (b_1 + b_2 + b_3)] \\
&= 300 - 10(a_1 + a_2 + a_3) + 30 - (b_1 + b_2 + b_3) \\
&= 330 - [10(a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)] \\
&= 330 - 200 = 130
\end{aligned}$$

	Given Nos		Digits replaced by 9 Complement	
	T	u	T	u
(b)	a_1	b_1	$(9 - a_1)$	$(9 - b_1)$
	a_2	b_2	$(9 - a_2)$	$(9 - b_2)$
	a_3	b_3	$(9 - a_3)$	$(9 - b_3)$
	$(a_1 + a_2 + a_3)$	$(b_1 + b_2 + b_3)$	$[27 - (a_1 + a_2 + a_3)]$	$[27 - (b_1 + b_2 + b_3)]$

Standard form of the sum of the 10 Complement digits numbers is

$$= 270 - 10(a_1 + a_2 + a_3) + 27 - (b_1 + b_2 + b_3)$$
$$= 297 - [10(a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)]$$
$$= 297 - 200 = 97$$

(60). Do as instructed in sum 59.

(61). 121, 132, ... 198, ... satisfy the condition
From 100 to 199, there are 36 numbers (if zero is not included–with zero included there $36 + 8 = 44$ numbers)
From 200 to 299, we have the following list
231, 232, 241, 242, 243, ... 291, 292, ... 298. Here there are $2 + 3 + 4 + \ldots 8 = 35$ (if zero is also included for units place it is $35 + 7 = 42$).
From 300 to 399, there are 33 (39).
From 400 to 499, there are 30 (35).
From 500 to 599, there are 26 (30).
From 600 to 699, there are 21 (24).
From 700 to 799, there are 15 (17).
From 800 to 899, there are 8 (9). Thus there are
$36 + 35 + 33 + 30 + 26 + 21 + 15 + 8 = 204$ numbers are there (where unit digit zero is not included). If zero is included in the

unit digit of the numbers, then there are 240 numbers. You can observe the pattern of numbers in each interval 100–199 200–299...800–899.

36, $(36-1)=35$, $(35-2)=33$, $\cdots$ $(15-7)=8$.

44, $(44-2)=42$, $(42-3)=39$, $(39-4)=35$, $\cdots$ $17-8=9$.

(62). Let the two odd numbers be $a=2m-1$ and $b=2n-1$.

$$a+b = 2m+2n-2 = 2(m+n-1).$$
$$a-b = (2m-1)-(2n-1) = 2(m-n).$$

The product is $2(m+n-1)\times 2(m-n)$
$$=4(m+n-1)(m-n)$$

If both m and n are even (or odd) $m-n$ is even. If one of m and n is odd, then $m+n-1$ is even. Thus, whatever be the parity (even or odd) of m & n, one of $m+n-1$ or $m-n$ is even. Thus $4(m+n-1)(m-n)$ is divisible by $4\times 2=8=2^3$.

(63). The first 4 prime numbers are 2, 3, 5 and 7. For a, b, c we can have 4 triplets (2, 3, 5) (2, 3, 7) (2, 5, 7) (3, 5, 7).
From each of the four triplets, we can get 6 arrangements totalling $4\times 6=24$ arrangements.
Eg. 2, 3, 5 2, 5, 3 3, 2, 5 3, 5, 2 5, 2, 3 5, 3, 2.
For this we get 6 different values for n, *i.e.*, $n=2^3\times 3^2\times 5$ or $2^3\times 5^2\times 3$ or $3^3\times 2^2\times 5$ or $3^3\times 5^2\times 2$ or $5^3\times 2^2\times 3$ or $5^3\times 3^2\times 2$.
Thus for all the four triplets, we get totally 24 different values. The least value of n is $2^3\times 3^2\times 5=360$. The biggest value of n is $7^3\times 5^2\times 3=25725$.

64). (a)

$$\frac{1}{1\cdot 2}=\frac{1}{1}-\frac{1}{2}$$

$$\frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3}$$

$$\vdots \qquad \vdots$$

$$\frac{1}{999\cdot 1000}=\frac{1}{999}-\frac{1}{1000}$$

Thus $1-\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\frac{1}{999}-\frac{1}{1000}$

$$=1-\left(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{999}-\frac{1}{1000}\right)$$

$$=1-\left(1-\frac{1}{1000}\right)$$

$$=\frac{1}{1000}$$

(b)

$$\left(1-\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\cdots\frac{1}{n\times(n+1)}\right)=\frac{1}{n+1}$$

$$=0.000001$$

$$i.e., \frac{1}{n+1}=\frac{1}{1000000}$$

$$n=1000000-1=999999.$$

(65). (a)

$$\frac{1}{2}=\frac{3}{6}=\frac{1+2}{6}=\frac{1}{6}+\frac{2}{6}=\frac{1}{6}+\frac{1}{3}$$

(b)

$$\frac{1}{4}=\frac{12}{48}=\frac{1+2+3+6}{48}=\frac{1}{48}+\frac{1}{24}+\frac{1}{16}+\frac{1}{8}$$

(c)

$$\frac{1}{6}=\frac{24}{144}=\frac{1+2+3+4+6+8}{144}$$
$$=\frac{1}{144}+\frac{1}{72}+\frac{1}{48}+\frac{1}{36}+\frac{1}{24}+\frac{1}{18}$$

Note: Express the given fraction into a suitable equivalent fractions; Express the numerator as sum of required number of natural numbers, so that each of the number is a divisor of the denominator.

(66). (b) $\frac{1}{3}=\frac{4}{12}=\frac{6-2}{12}=\frac{1}{2}-\frac{1}{6}$

(c) $\frac{1}{4}=\frac{3}{12}=\frac{4-1}{12}=\frac{1}{3}-\frac{1}{12}$

$$=\frac{6}{24}=\frac{12-6}{24}=\frac{1}{2}-\frac{1}{4}\cdots$$

Chapter 10

Divisors and Multiples: lcm and gcd

The number of divisors of a natural number $N = p_1^{\alpha_1} \times p_2^{\alpha_2} \cdots \times p_n^{\alpha_n}$ is $(\alpha_1 + 1)(\alpha_2 + 1)\ (\alpha_3 + 1)\cdots(\alpha_n + 1)$, where $p_1, p_2, \cdots p_n$ are all distinct prime numbers.
Sum of the divisors of N

$$= \frac{(p_1^{\alpha_1+1} - 1)}{(p_1 - 1)} \times \frac{(p_2^{\alpha_2+1} - 1)}{(p_2 - 1)} \times \cdots \frac{(p_n^{\alpha_n+1} - 1)}{(p_n - 1)}$$

Sum of the reciprocal of the divisors of N

$$= \frac{(p_1^{-(\alpha_1+1)} - 1)}{(p_1^{-1} - 1)} \times \frac{(p_2^{-(\alpha_2+1)} - 1}{(p_2^{-1} - 1)} \times \cdots \frac{(p_n^{-(\alpha_n+1)} - 1)}{(p_n^{-1} - 1)}$$

Examples: $N = 6 = 2^1 \times 3^1$ gives the number of divisor as $2 \times 2 = 4$ (1, 2, 3, and 6).
$N = 28 = 2^2 \times 7^1$ gives the number of divisors as $(2+1) \times (1+1) = 6$ (1, 2, 4, 7, 14, 28 are the divisors of 28).
$N = 48 = 2^4 \times 3^1$ gives $(4+1) \times (1+1) = 10$ divisors. Sum of the

divisors of 6 is

$$\frac{3^{1+1}-1}{(3-1)} \times \frac{2^{1+1}-1}{(2-1)} = \frac{8 \times 3}{2 \times 1} = 12$$

(and 1+2+3+6=12)
Sum of the divisors of 28 is

$$\frac{2^{2+1}-1}{2-1} \times \frac{7^{1+1}-1}{7-1} = \frac{7}{1} \times \frac{48}{6} = 56.$$

Sum of the divisors of 48 is

$$\frac{2^{4+1}-1}{2-1} \times \frac{3^{1+1}-1}{3-1} = \frac{31}{1} \times \frac{8}{2} = 124.$$

$(1+2+3+4+6+8+12+16+24+48 = 124!)$
Sum of reciprocal of the divisors of 6 =

$$\begin{aligned} 2^1 \times 3^1 &= \frac{2^{-(1+1)}-1}{(2^{-1}-1)} \times \frac{3^{-(1+1)}-1}{(3^{-1}-1)} \\ &= \frac{\frac{1}{4}-1}{\frac{1}{2}-1} \times \frac{\frac{1}{9}-1}{\frac{1}{3}-1} \\ &= \frac{-3}{4} \times \frac{2}{-1} \times \frac{-8}{9} \times \frac{3}{-2} = 2. \end{aligned}$$

Apply the formula and find the sum of the reciprocals of 28 and 48 also

(1). (a) $N = 2^4 \times 31^1 (= 496)$. No of divisors $5 \times 2 = 10$.
Sum of the divisors

$$\begin{aligned} &= \frac{2^5-1}{2-1} \times \frac{31^2-1}{31-1} \\ &= \frac{31}{1} \times \frac{960}{30} = 31 \times 32 = 2(496) = 2N \end{aligned}$$

The divisors are 1, 2, 2^2, 2^3, 2^4, 31, 2×31, $2^2 \times 31$, $2^3 \times 31$, $2^4 \times 31$. Sum of the reciprocal of the divisors

$$= \frac{2^{-5}-1}{2^{-1}-1} \times \frac{31^{-2}-1}{31^{-1}-1}$$
$$= \frac{31}{32} \times \frac{2}{1} \times \frac{960}{961} \times \frac{31}{30} = 2$$

Note: The numbers 6, 28, $2^4 \times 31 = 496 \ldots$ are perfect numbers as the sum of the divisors of these numbers is twice the given number and the sum of the reciprocals of these numbers are always 2.

(b) Do the other subdivisions

(2). Do it yourself

(3). If $a_1 a_2 a_3 \cdots a_{2n}$ is a $2n$ digit number divisible by 11, a_1, a_2, $\cdots a_{2n}$ are the digits of N.

$$A = (a_1 + a_3 + \cdots a_{2n-1}) \sim (a_2 + a_4 + \cdots a_{2n})$$

is a multiple of 11 (Divisibility test by 11). The number got by reversing the digits is $a_{2n}\ a_{2n-1}\ a_{2n-2} \cdots\ a_3\ a_2\ a_1 = N'$ (say) Now $(a_{2n} + a_{2n-2} + \cdots a_2) \sim (a_{2n-1} + a_{2n-3} + \cdots a_3 + a_1)$ is the same as A. Thus if N is divisible by 11, then the number N' got by reversing the digits of N is also divisible by 11. If $a_1\ a_2\ a_3 \cdots a_{2n-1}$ is a $2n-1$ digit number, let

$$N = (a_1 + a_3 + \cdots a_{2n-1}) \sim (a_2 + a_3 + \cdots a_{2n-2})$$
$$= 0 \quad \text{or a multiple of 11.}$$

So, reversing the digits, we get

$$N' = (a_2 + a_3 + a_4 + \cdots a_{2n-2}) \sim (a_1 + a_3 + \cdots a_{2n-1})$$

is also 0 or the same multiple of 11 as in the number N. Thus if $a_1\ a_2\ a_3 \cdots a_{2n-1}$ is divisible by 11, so also, the number got by reversing the digit is divisible by 11.

(4). $4a3b$ is divisible by 11.

$$\therefore (4+3) \sim (a+b) = 0 \text{ or } 11 \text{ or } 22$$
$$i.e., \quad 7 \sim (a+b) = 0$$
$$\text{say} \quad a+b-7 = 0$$
$$\text{giving} \quad a+b = 7.$$

then the pairs (a, b) are

$$(0,7), (1,6), (2,5), (3,4), (4,3), (5,2), (6,1), (7,0) \qquad (1)$$

i.e., The number $4a3b$, where a, b are replaced by any of the pairs (a, b) got in Eqn. (1) is divisible by 11.

$$\text{If} \quad (4+3) \sim (a+b) = 11$$
$$(a+b) - 7 = 11$$
$$a+b = 18, \; a, \; b \text{ are single digit nos.}$$
$$[\text{or} \quad 7-(a+b) = 11, \quad \text{is not possible}]$$

$a = b = 9$ is the only possible value. So let $(a, b) = (9, 9)$. $(a+b) - 7 = 22 \Rightarrow a+b = 29$. There is no single digit numbers a, b satisfying $a+b = 29$. So no more answers are possible. Thus there are 9 values for the pairs (a,b). Thus there are 9 numbers with thousands place 4 and 10s place 3, such that they are divisible by 11. Using $(a, b) = (1, 6)$, we get (1) 41 36 209 (2) 41 36 902 (3) 63 14 209 (4) 63 14 902 (5) 14 63 209 are 7 digit numbers divisible by 11.

(5). Let us first list all two digit numbers, which are divisible by their units digit; Then, from among these numbers let us count, how

many are also divisible by their 10s place.

unit digit 1

(1)11(2) 21(3)31(4)41,...(9)91 9 numbers

unit digit 2

(1)12(2) 22(3) 32(4) 42,...(9) 92 9 numbers

unit digit 3

(1)33(2) 63(3) 93 (use divisibility by 3.) 3 numbers

unit digit 4

(1)24(2) 44(3) 64(4) 84, 4 numbers

unit digit 5

(1)15(2) 25(3) 35(4) 45,...(9) 95 9 numbers

unit digit 6

(1)36(2) 66(3)96 3 numbers

unit digit 7

(1)77 1 number

unit digit 8

(1)48(2) 88 2 numbers

unit digit 9

(1)99 1 number

Total number of 2 digit numbers which are divisible by their units digit is 9+9+3+4+9+3+1+2+1=41 numbers of the above listing (and classification)

In the first row a, we have just one number 11, divisible by 10s place also thus we have one number in the first row divisible by both units and 10s place. In the second row, we have 12 and 22 divisible by their 10s digit also.

Thus we have two numbers in the second row divisible by both units and 10s place. Similarly in 3^{rd}, 4^{th},... 9^{th} rows we find 1, 2,

2, 2, 1, 2 and 1 numbers respectively, divisible by 10s and units place numbers. Thus we have,

$$1+2+1+2+2+2+1+2+1=14.$$

Thus there are fourteen 2 digit numbers which are divisible by both their units and terms digits.

Note: A two digit number ab (with units place a and tens place b) in standard notation is $10a+b$. Thus $b \mid 10a+b$, means, $b/10a$ as b always divides b and so, it is left to see if $b/10a$. [$b \mid 10a$, read b divides $10a$] $b \mid (10a+b)$ means $b \mid 10a$, as b always divides b. So it is left to see if $b \mid 10a$; Similarly if $a \mid 10a+b$ then $a \mid b$, as a always divides $10a$. So it is left to see if $b \mid 10a$; read $a \mid b$ as 'a divides b.'

If $d \mid (a+b)$ it is not necessary that it should divide both a and b.

$$5 \nmid (18+7) \quad \text{but} \quad 5 \mid 18, 5 \nmid 7$$

So whenever $d \mid a+b$, if $d \nmid a$; then $d \nmid b$ also, and $d \mid a+b$ and if $d \mid a$, then $d \mid b$ also.

(6). a) The units and tens places are a and b. Thus $10a+b$ is the number.

We want $(a+b)$ to divide $10a+b$ or $a+b$ to divide $(9a)+(a+b)$. Since $(a+b) \mid (a+b)$ we should find values of a and b, such that $(a+b)/9a$.

We fix values for b, find the corresponding values of a.

Remember $0 < a \leq 9$.

$$b=0 \quad \Rightarrow \quad a+b=a$$

and $a+b/9a$ means $a/9a$ which is true for all a, $\quad 0 < a \leq 9$. Thus the 2 digit numbers 10, 20, 30,... 90, the sum of the digits divides the number.

b	$a+b$	$a+b \mid 9a$	Values of a	Nos.
1	$a+1$	$a+1 \mid 9a$	$a+1=3$ or	$21, 81$
			$a+1=9$	
			$\Rightarrow a=2$	
2	$a+2$	$a+2 \mid 9a$	$a+2=3$	$12, 42, 72$
			$a+2=6$	
			$a+3=9$	
3	$a+3$	$a+3 \mid 9a$	$a+3=9$	63
			$a=6$	
4	$a+4$	$a+4 \mid 9a$	$a=2$	$24, 54, 84$
5	$a+4$	$a+5 \mid 9a$	$a=4$	45
6	$a+6$	$a+6 \mid 9a$	$a=3$	36
7	$a+7$	$a+7 \mid 9a$	$a=2$	
8	$a+8$	$a+8 \mid 9a$	$a=1,4$	$18, 48$
9	$a+9$	$a+9 \mid 9a$	$a=0$	
		but a being in tens place is not $=0$		

Thus the two digit numbers that are divisible by their digital sum is 21, 81, 12, 42, 72, 63, 24, 54, 84, 45, 36, 18, 48, 10, 20, 30, 40, 50, 60, 70, 80, 90.

b) For product of the digits dividing numbers do it yourself.

(7). The two digit premium numbers has both the digits prime. The single digit prime numbers are 2, 3, 5 and 7.

22	32	52	72
$\boxed{23}$	33	53	73
25	35	55	75
27	$\boxed{37}$	57	77

So the premium numbers are thus there are 16 premium numbers and the circled premium numbers 23, 37, 53 and 73 are prime premium numbers.

(8). Sum of two prime numbers are not always a prime number. In fact the sum of two odd primes are always even and hence composite. However there are *infinitely many* prime numbers

which are the sum of 2, the only even prime and an odd prime! 11+2=13, 17+2=19, 41+2=43, 29j+2=3, and so an.

(9). 100=3+97, =11+89, =17+83, =41+59, =47+53,

(10). 47□ × 3□2 = 162792, to replace the boxes by the suitable digits, factorise 162 792;

$$162792 = 2^3 \times 7 \times 3^2 \times 17 \times 19$$
$$2 \times 3^2 \times 19 = 2 \times 9 \times 19 = 342$$
$$\text{and} \quad 2^2 \times 7 \times 17 = 476$$

Thus $47\boxed{6} \times 3\boxed{4}2 = 2^3 \times 7 \times 3^2 \times 17 \times 19 = 162792$. and the boxes are to be replaced by 6 and 4.

(11). By division algorithm
$75 \times 72 + 72 =$ Dividend.
$= 72(75 + 1) = 72 \times 76$, is the given number.
$\therefore$ Given number $\div$ 72
gives $75 + 1 = 76$ Quotient
remainder *Zero*

(12). Do it as before.
Hint: 9778 × 9778 + 9782
= 9778 × 9778 + 9778 + 4
=9778 × [9778 + 1] + 4.

(13). The short-cut for division by 9997 = 10000-3.

	Quotient	Remainders
Q_1	979	7067 +979 × 3 $R_1 = 10004 > 9997$.
Q_2	$(10004 \div 10000) = 1$	$R_2 = 4 + 1 \times 3 = 7$.

So, the actual quotient is 980 and the actual remainder is 7. Give reason for your calculation.

Hint: 9997 is 3 less than 10000 and so the actual remainder is 3 $\times$ q (got by dividing the number by 10000) + remainder (got by dividing the number by 10000).

(14). Since 727, 72727, 7272727 ... are neither divisible by 2 nor by 3, and hence you should consider only the factors $72 \times 7272 \times 727272 \times 72727272$.

$$\begin{aligned}
&= 6^2 \times 2 \times 6^2 \times 202 \times 6^2 \times 20202 \times 6^2 \times 2020202 \\
&= 6^2 \times 6^2 \times 6^2 \times 6^2 \times 2 \times 202 \times 20202 \times 2020202 \\
&= 6^8 \times 6 \times \underline{3367 \times 2 \times 202 \times 2020202} \\
&= 6^9 \times 3367 \times 2 \times 202 \times 2020202.
\end{aligned}$$

Since 3367 is neither divisible by 2 nor by 3, 6 cannot be a factor of 3367 and 2, 202, 2020202 are not divisible by 3, cannot give 6 as a factor.

Thus 9 is the highest power of 6 dividing the given number.

(15). Separate the power of factors of 7 and 4 (in factor power of 2).

$$7 \times 14 \times 21 \times 28 \times 35 \times 42 \times 49 \cdots \times 98$$

49 and 98 are the numbers divisible by 7^2. Thus the highest power of 7, in the product is $14 + 2 = 16$.

The highest power of 2, should be counted in the even multiple of 7. There 7 even multiples. And every 4^{th} multiple of 7 is divisible by 4, or 2^2.

Thus 28, 56, 84 are multiplies 2^2 of which 56, (the 8^{th} multiple) is a multiple of 2^3.

Thus the highest power of 2 in the product is $7 + 3 + 1 = 11$.

where K is the product of some numbers other than 2 and 7 occurring in the given product, then

$$\text{the given product of numbers} = 2 \times (2^2)^5 \times 7^5 \times 7^{11} \times K.$$

$$= 4^5 \times 7^5 \times 2 \times 7^{11} \times K.$$
$$= 28^5 \times 2 \times 7^{11} \times K.$$

Thus the highest power of 28, dividing the number is 5.

(16). Refer Problem 46 in Chapter 1.

(17). Refer Problem 47 in Chapter 1.

(18). Refer Problem 48 in Chapter 1.

(19). Refer Problem 49 in Chapter 1.

(20). See the next problem.

(21). The tens place of 1! +2! +3! + ... + 100!
From 10! onwards the tens place of the number is zero. So, it is enough if units and tens places of numbers from 1! to 9! are added.
So $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9!$

$$\begin{aligned} &= 1 + 2 + 6 + 24 + \boxed{1}\,20 + \boxed{7}\,20 \\ &\quad + \cdots 40 + 20 + 80 + \cdots + 40 + \cdots 20 + \cdots 80 \\ &= \cdots 13. \end{aligned}$$

So the tens place and units place of the sum of the numbers is 1 and 3 respectively.

(22). (a) $4 \times 5 \times 6 = n!$ and $1 \times 2 \times 3 = 6$
$\therefore 4 \times 5 \times 6 = 4 \times 5 \times (1 \times 2 \times 3) = 5! = n!$
$\therefore n = 5$

(b) $4 \times 5 \times 6 = \dfrac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 2 \times 3} = \dfrac{6!}{3!}$

(c) $60 = \dfrac{120}{2} = \dfrac{5!}{2!}$

(23). The three numbers are
$75!$, $2^{37}\times 3^{25}\times 5^{15}\times 7^{10}\times 11^{6}$ and $13^{5}\times 17^{4}\times 19^{3}\times 23^{3}\times 29^{2}\times 31$. The highest power of the prime numbers 2, 3, 5, 7 and 11, dividing $75!$ is

$$\left[\frac{75}{2}\right]+\left[\frac{75}{4}\right]+\left[\frac{75}{8}\right]+\left[\frac{75}{16}\right]+\left[\frac{75}{32}\right]+\left[\frac{75}{64}\right]+0+\cdots$$

$$37+18+9+4+2+1=71,$$

$$\left[\frac{75}{3}\right]+\left[\frac{75}{9}\right]+\left[\frac{75}{27}\right]+\left[\frac{75}{81}\right]=25+8+2+0=35.$$

$$\left[\frac{75}{5}\right]+\left[\frac{75}{25}\right]+\left[\frac{75}{125}\right]=15+3+0=18.$$

$$\left[\frac{75}{7}\right]+\left[\frac{75}{49}\right]=10+1=11,$$

$$\left[\frac{75}{11}\right]=6$$

Thus $2^{71}\times 3^{35}\times 5^{18}\times 7^{11}\times 11^{6}$ divides $75!$, with quotient > 0, and since,

$$2^{71}\times 3^{35}\times 5^{18}\times 7^{11}\times 11^{6} > 2^{37}\times 3^{25}\times 5^{15}\times 7^{10}\times 11^{6}$$

$$75! > 2^{37}\times 3^{25}\times 5^{15}\times 7^{10}\times 11^{6}.$$

Similarly find the greater of $75!$ $13^{5}\times 17^{4}\cdots\times 31$ and then compare 2rd and 3rd numbers and complete the problem.

(24). If 5 divides a square number x^2 (*i.e.*, x^2 is divisible by 5) where $x \in \mathbb{N}$ then 25 divides x^2
We shall show here, that only x is a multiple of 5, x^2 is a multiple of 5 and then show that x^2 is a multiple of 25. If $x \neq 5k$, *i.e.*, x is not divisible by 5 let us take $x = 5k + r$, $0 < r < 5$,

$x^2 = 25k^2 + 10kr + r^2$.
Since r can take the values 1, 2, 3 and 4 we get

$$\begin{aligned} x^2 = 25k^2 + 10k + 1 &= 5(5k^2 + 2k) + 1. \\ 25k^2 + 20k + 4 &= 5(5k^2 + 4k) + 4. \\ 25k^2 + 30k + 9 &= 25k^2 + 30k + 5 + 4 \\ &= 5(5k^2 + 6k + 1) + 4. \\ 25k^2 + 40k + 6 &= 25k^2 + 40k + 15 + 1 \\ &= 5(5k^2 + 8k + 3) + 1. \end{aligned}$$

i.e., x^2 is not divisible by 5 if x is not divisible by 5, as x^2 leaves remainders, 1, 4, 4, 1 (other wise). Therefore x^2 is divisible by 5 only if x is divisible by 5.
Let $x = 5k$. $x^2 = x \times x = 5k \times 5k = 25k^2 = 5 \times 5k^2$. Thus x^2 is divisible by 5, if x is divisible by 5. Again $x^2 = 5 \times 5k^2 = 25 \times k^2$, thus x^2 is divisible by 25. [Note that this proof is valid only for prime numbers dividing x^2].
Example: 12 divides x^2 does not imply $144 \mid x^2$, but it implies that $36 \mid x^2$. Check.
Example: $12 \mid 324 = 18^2$, but $12^2 = 144 \nmid 324$ however $36 \mid 324$ as $324 = 36 \times 9$.

(25). Take the numbers $n_1, n_2 \cdots$ be $3k$, $3k+1$, $3k+2$, $4k$, $4k+1$, $4k+2$, $4k+3$, $\cdots$ and square each of these numbers, find the remainders on division by $3, 4, \cdots$ [Any natural number is the form $3k$, $3k+1$, $3k+2$, or $4k$, $4k+2$, $4k+3$, $\cdots$]

(26). Cube the above numbers and find the remainders.

(27). Let $a = 13$, $b = 12$ then $a + b = 25$, and $5 \nmid 13$, $5 \nmid 12$, but $5 \mid 25$. The pairs (a, b) in this case are (11, 14), (11, 19), (12, 13), (12, 18), (14, 16), (17, 13)
Find such (a, b) for other subdivisions also.

(28). $a \times b \mid 48$. Thus $a \times b$ is a non unit divisor of 48 *i.e.*, $a \times b$ is one of 2, 3, 4, 6, 8, 12, 16, 24, 48. (a, b) are non relatively prime. So, we cannot take $a \times b$ as 2, 3 or even 6.

$$(a,b) = (2,2) \qquad \text{so that } a \times b = 4$$
$$(a,b) = (2,4),\ (4,2) \qquad \text{so that } a \times b = 8$$
$$(a,b) = (2,6),\ (6,2) \qquad \text{so that } a \times b = 12$$
$$(a,b) = (4,4) \quad \text{or} \quad (2,8),\ (8,2) \qquad \text{so that } a \times b = 16$$
$$(a,b) = (2,12),\ (12,2) \text{ or } (6,4),\ (4,6) \qquad \text{so that } a \times b = 24$$
$$(a,b) = (2,24),\ (4,12),\ (6,8) \qquad \text{so that } a \times b = 48$$
$$(24,2),\ (12,4),\ (8,6)$$

Thus there are 18 pairs of (a, b).

(29). *Hint:* $a \times b$ must be a multiple of 48 say 48, 48×2, $48 \times 3, \cdots$ a is such that $10 \le a \le 30$, and b is such that $10 \le b \le 100$. Two of the answers are.
Taking $a = 11, b = 48$, we get $48 \mid 11 \times 48$. or taking $a = 13$, $b = 16$ we get $48 \mid 13 \times 16$ etc.
Find at least 5 such answers.

(30). Refer the result of Problem 29 and proceed.

(31). *Hint:* $n = 3k + 2$; find n^2 and n^3.

(32). (c) gcd of 2 numbers $a, b \le 200$ is 60.
Each of a, b must be a multiple of 60.
Thus (120, 180) is such a pair. Do the other subdivision.

(33). gcd of 120 and a is 40. $a < 200$. a should be a multiple of $40 = 2^3 \times 5$. But as $120 = 2^3 \times 5 \times 3$, it should not be a multiple of 3. Thus $a = 40 \times 1$, 40×2 or 40×4. Thus (120, 40), (120, 80), (120, 160) has gcd 40, as $a > 50$, (120, 80) and (120, 100) are the 2 pairs and the corresponding values of a are 80, 60.

(34). lcm of 2 numbers say a, b is 224.
Both a and b are the divisors of 224.
or 224 is a common multiple of a and b and also the least of such common multiple. Thus we find the divisors of 224

$$224 = 32 \times 7 \quad \text{or} \quad 2^5 \times 7.$$

We can find two numbers, so that at least one of these should have 2^5or7 as a divisor. Thus one number is $2^3 \times 7$ and the other number is 2^5
Similarly you can have $2^3 \times 7, 2^5 \times 7, 7, 2^5$, etc.
You can start with $(7, 2^5)$ and get the pairs $(7\times2, 2^5)$, $(7\times2^2, 2^5)$, $(7 \times 2^3, 2^5)$, $(7 \times 2^4, 2^5)$, $(7 \times 2^5, 2^5)$, $(7 \times 2^5, 2^4)$, $(7 \times 2^5, 2^3)$, $7 \times 2^5, 2^2)$. $(7 \times 2^5, 2)$ are the pair of numbers whose lcm is $7 \times 2^5 = 224$.

(35). Do it yourself.

(36). Do it yourself.

(37). The common multiple of 15 and 24 is divisible by both 15 and 24.
So, the common multiple +3 gives the required result. The numbers are

$$\begin{aligned} &120 + 3 \quad (\text{lcm of 15, 24 is 120}) \\ &240 + 3 \\ &360 + 3 \quad \text{and so on,} \end{aligned}$$

the smallest number being 123.

(38). Using Division algorithm we have $7k + 5 = N = 15l + 7$ for sum $k, l \in N$.

$$\text{So} \quad 7k + 5 = 15l + 7$$

$$\Rightarrow 7k = 15l + 2$$
$$= 14l + (l + 2)$$
$$\Rightarrow k = \frac{14l}{7} + \frac{l+2}{7}$$
$$= 2l + \frac{l+2}{7}$$
$$k = 2l + \frac{l+2}{7}$$

and choose value for l, so that $\frac{l+2}{7}$ is a natural number. We shall prepare a tabular column.

l	5	12	19	...
k	11	26	41	...

Thus the numbers N are $15l + 7 = (7k + 5)$ is 82, 187, 292, 397, ...

Note: 5, 12, 19, ... is a sequence numbers with difference between consecutive terms 7. It is an arithmetic progression (AP). Again 11, 26, 41, ... is also a sequence of numbers with difference between consecutive terms 15 (AP with common difference 15 and 82, 187, 292, ...) is again an AP with common difference the lcm of 7 and 15 (to this case it is $7 \times 15 = 105$)

(39). Let us take $N = 6k + 4$, for $k \in W$. Now, we shall investigate remainder dividing $6k + 4$, for $0 \leq k \leq 6$ (since $k = 6$ gives $N = 36 + 4$)
and for $k = 6$, $N > 36$, for $k \leq 6$, $N < 36$.

k	0	1	2	3	4	5	6
N	4	10	16	22	28	34	40
Remainder when N is $\div$ by 36	4	10	16	22	28	34	4

You many now find that for $k = 7, 8, 9, \ldots$, the remainders are 10, 16, 22, 28, and 34, ... Repeat.
Thus when N, which leaves a remainder 4 on dividing by 6, leaves any one of the remainders 4, 10, 16, 22, 28, 34, ... when divided by $6^2 = 36$.
Note: 4, 10, 16, ... is an arithmetic progression with common difference 6 and 4 is the first term, which is the given remainder when N is divided by 6.

(40). Given that 2 divides N $2^2 = 4 \nmid N$, also $2^5 = 32 \mid (n-6)$, but $2^6 \nmid (n-6)$
Let $n - 6 = k \times 2^5$, where k is an odd number.
$n + 2 = n - 6 + 8 = 2^5k + 8 = 2^5k + 2^3 = 2^3(4k+1)$
so $(n+2)$ is divisible by $8 = 2^3$ but not by $16 = 2^4$ as $(4k+1)$ is an odd number.
$n + 6 = n - 6 + 12 = 2^5k + 12 = 2^2[2^3k + 3]$
Thus $2^2 | (n+6)$, but $2^3 \nmid (n+6)$

(41). 3^4 divides n and $3^5 \nmid n$.

$$\begin{aligned} \therefore n &= (3l \pm 1)3^4, \quad \text{where } l \text{ is a natural number.} \\ n &= 3^5 l \pm 3^4 \\ n - k &= (3^5 l \pm 3^4) - k \\ &= 3^2(27l \pm 9) - k \end{aligned}$$

Since $(n-k)$ is divisible by 3^2, k should be $9 = 3^2$.

$$\begin{aligned} \text{Thus} \quad (n-k) &= 3^5 l \pm 3^4 - 3^2 \\ &= 3^2[3^3 l \pm 3^2 - 1] \end{aligned}$$

is divisible by 3^2 but not by 3^3.
$3^3 \mid n + m$ but $3^4 \nmid (n+m)$,
$n = 3^5 l \pm 3^4 = 3^5 l \pm 81$.
Since 3^3 should divide $n + m$, but $3^4 \nmid n + m$

$$3^3 \mid 3^5 l \pm 81 + m.$$

If m is 27, we get

$$\begin{aligned} m+n &= 3^5 l \pm 81 + 27 \\ &= 3^5 l + 108 \quad \text{or} \quad 3^5 l - 54, \end{aligned}$$

$$\text{in either case,} \quad 3^3 \mid (m+n), \quad \text{but} \quad 3^4 \nmid (m+n)$$

(42). The four digit numbers to be found is of the form $2005k + 4$

k	1	2	3	4
$N = 2005k + 4$	2009	4014	6019	8024

For $k = 5$, $2005k + 4$ becomes a 5 digit number. So there first four values for n.

(43). The lcm of 2 numbers is 2000, thus both the numbers divides 2000.

$$2000 = 2^4 \times 5^3$$

So, each of the numbers should have as their factors. $2^n \times 5^k$ when $n \leq 4$ and $k \leq 3$ and such that each of the numbers is a 3 digit number and at least one of numbers should have 5^3 and 2^4. $5^3 = 125$ is a 3 digit number.
The pairs are

$$\begin{array}{cc} 5^3 & 5^2 \times 2^4 \\ 5^3 \times 2 & 5^2 \times 2^4 \\ 5^3 \times 2^2 & 5^2 \times 2^4 \end{array}$$

Since $5^3 \times 2^3$ and $5^3 \times 2^4$ are 4 digit numbers and 5×2^4 is a two digit number, we cannot have other combinations. Thus there are 3 pairs of numbers (125, 400), (250, 400), (500, 400)

(44). Product of 5 of the numbers 2, 3, 5, 7, 9 and 11 is 2310. Since the digital sum is 6, it is not divisible by 9 and thus 9 is left out and $2310 = 2 \times 3 \times 5 \times 7 \times 11$.
Now, we want to find $2 \times 3 \times 7 \times 11 \times 9$

$$= \frac{2 \times 3 \times 5 \times 7 \times 11}{5} \times 9$$
$$= \frac{2310}{5} \times 9 = 462 \times 9 = 4158.$$

45). Using all the prime number s 2, 3, 5, 7, the number of 1 digit number is (trivially) the same four numbers 2, 3, 5, 7. Now let us count the two digit numbers

Case I(if repetition is not allowed)

23	32	52	72
25	35	53	73
27	37	57	75

Thus there are 12, 2 digit numbers where repetition is not allowed.

4 digits are used, we have 12, units digits and 12 tens digits; all the four digits 2, 3, 5, 7 have equal chances. Thus, each digit occurs three times in units place and 3 times in tens place. So the sum of these 12 numbers is

$$= 30(2+3+5+7) + 3(2+3+5+7)$$
$$= 30 \times 17 + 3 \times 17 = 33 \times 17 = 561.$$

For 3 digit numbers, [we will have $4 \times 3 \times 2 = 24$ numbers]. For each 3 digit number we leave one of the four digits, By arranging the 3 digit chosen in each case, we get 6 different numbers. Digits 2 3 5 gives (1) 235, (2) 253, (3) 352, (4) 325, (5) 523, (6) 532.

Thus, we get group of 3 digits, 4. *times*, as each time we leave one digit. So, the total numbers got would be 24.

As before each of the 4 digits have equal chance of appearing in each of the places (units, tens and hundreds), each digit appears $\frac{24}{4} = 6$ times, so the sum of the 3 digit numbers is

$$600(2+3+5+7)+60(2+3+5+7)$$
$$+6(2+3+5+7) = 666 \times 17 = 11322$$

Likewise do for 4 digit numbers also [Again there are 24 numbers as, the 4 digit can be arranged to get 24 different 4 digit numbers]. For 5 digit , 6 digit, ... we should repeat the digits. Thus numbers with more than 4 digits are possible (only if repetition is allowed).

We shall discuss the numbers when repetition is allowed, from 2 digit numbers on words.

Case II (When repetition is allowed)
2 digit numbers are $4 \times 4 = 16$

22	32	52	72
23	33	53	73
25	35	55	75
27	37	57	77

The sum of the numbers is
$40 \times 17 + 4 \times 17 = 680 + 68 = 748 = 44 \times 17$. For 3 digit numbers, since there 4 digits. Total number of 3 digit numbers becomes $4 \times 4 \times 4 = 64$. [For each digit fixed for 100s place, there 4 different digits available for 10s place, again for each 10s place there 4 different digit available for units place, then for any fixed digit in 100s place there $16 = 4 \times 4$ different 3 digit numbers

available for 10s and units place. Thus for 4 different digits for 100s place there are $4 \times 16 = 64$ different 3 digit numbers can be got]

The sum of the numbers

$$\frac{6400}{4} \times 17 + \frac{640}{4} \times 17 + \frac{64}{4} \times 17 \quad \text{why?}$$
$$= 1600 \times 17 + 160 \times 17 + 16 \times 17$$
$$= (1776) \times 17 = 30192$$

you can see that there are $4^4 = 256$ four digit numbers.... In general there are 4^n, n digit numbers. The sum of these n digit numbers is

$$4^{n-1} \times 10^{n-1} \times 17 + 4^{n-2} \times 10^{n-2} \times 17 \cdots$$
$$4^{n-1} \times 10 \times 17 + 4^{n-1} \times 17$$
$$= (10^{n-1} + 10^{n-2} + \cdots + 10 + 1) \times 4^{n-1} \times 17$$
$$= \underbrace{111 \cdots 111}_{n \text{ ones}} \times 4^{n-1} \times 17$$

(46).

$$14 \star \star 5 \div 2 \star 3 \star = 5$$

We have to replace the 4 stars by 6, 7, 8 and 9, as the digits 1 to 9 are used in the left.

$$14 \underset{3}{\star} \underset{4}{\star} 5 \div 2 \underset{1}{\star} 3 \underset{2}{\star} = 5 \Rightarrow 14 \underset{3}{\star} \underset{4}{\star} 5 \div 5 = 2 \underset{1}{\star} 3 \underset{2}{\star}$$
$$\Rightarrow 2 \underset{1}{\star} 3 \underset{2}{\star} \times 5 = 14 \underset{3}{\star} \underset{4}{\star} 5$$

$\star$ marked 2, should be an odd number either 7 or 9. If we take $\star$ marked 2 as 9

$$2 \underset{1}{\star} 39 \times 5 = \text{-- -- --} 95,$$

thus 2 nines are used one for $\star$ marked 2, and another for $\star$ marked 4. So $\star$ marked 2 should be 7.

$$2\underset{1}{\star}37\times 5 = \_\,\_\,\_\,\_\,85,$$

we get $\star$, marked 4 to be 8. Thus $2\underset{1}{\star}37 \times 5 = 14\underset{3}{\star}85$. So, the remaining numbers 6 and 9 to be fixed for $\underset{1}{\star}$ and $\underset{3}{\star}$. If $2\,6\,3\,7 \times 5$ is considered we get

$2\,6\,3\,7 \times 5 = 1\,3\,\underset{3}{\underline{1}}\,8\,5$, but $\underset{3}{\star}$ takes value 1, which is not true so we can consider

$$2937 \times 5 = 14\underline{6}85$$

Thus $\star$ takes the digit 9, $\underset{2}{\star}$ takes the digit 7, $\underset{3}{\star}$ takes the value 6 and $\underset{4}{\star}$ takes the value 8.

(47). 20 cm $\times$16 cm $\times$12 cm is the measurement of the cuboidol box. To fill it by number of cubical boxes, the edge of the cube must be the gcd of 20, 16 and 12cm, which is 4cm. Along lengthwise we can keep $\frac{20}{4} = 5$ cubes, breadth-wise $\frac{16}{4} = 4$ cubes and height-wise $\frac{12}{4} = 3$ cubes can be placed. Thus $5 \times 4 \times 3 = 120$ cubes of edge 4cm can be packed and thus the biggest cubes will have its edge 4cm. In this case we get the minimum number of cubes. If each cube has unit edge then we can pack $20\times 16\times 12\times = 3840$ cubes of integer edge and this the maximum number cube with minimum length for the edge to be 1 cm (integer length).
Note: If the edge can be a fraction then, the number of cuber increases when the edge is smaller and smaller.

(48). If the measurements of integer length are taken as $7\times 11\times 13$. Then along the length wise side we can have 5-cuboids, breath wise 3 cuboids and height wise 2 cuboids. Thus, the number of which of integer measurements $(7\times 11\times 13)$ will be $5\times 3\times 2 = 30$ cuboids. This $7 \times 11 \times 13$ is the dimension of the biggest integer

sided cuboids, the number of cuboids being the minimum.
The other measurements can be

$5 \times 11 \times 13$ $(7 \times 3 \times 2 = 42$ cuboids)
$5 \times 3 \times 13$ $(7 \times 11 \times 2 = 154$ cuboids)
$5 \times 11 \times 2$ $(7 \times 3 \times 13 = 273$ cuboids)
$5 \times 3 \times 2$ $(7 \times 11 \times 13 = 1001$ cuboids)
$7 \times 11 \times 13$ (already given)
$7 \times 11 \times 2$ $(5 \times 3 \times 13 \neq 195$ cuboids)
$7 \times 3 \times 13$ $(5 \times 11 \times 2 = 110$ cuboids)
$7 \times 3 \times 2$ $(5 \times 11 \times 13 = 725$ cuboids)

If the dimension are $1 \times 1 \times 1$ (cubes) the numbers of cubes that could be packed is $35 \times 33 \times 26 = 30030$.

(49). You can construct a multiplication table of this numbers and delete product repeated. You can now count the distinct product any of the two numbers., which give distinct value.

x^2	2^2	2^3	2^4	2^5	2^6	3^2	3^3	3^4	3^5
2^2	2^4	2^5	2^6	2^7	2^8	$2^2 \times 3^2$	$2^2 \times 3^3$	$2^2 \times 3^4$	$2^2 \times 3^5$
2^3	$\cancel{2^5}$	$\cancel{2^6}$	$\cancel{2^7}$	$\cancel{2^8}$	2^9	$2^3 \times 3^2$	$2^3 \times 3^3$	$2^3 \times 3^4$	$2^3 \times 3^5$
2^4	$\cancel{2^6}$	$\cancel{2^7}$	$\cancel{2^8}$	$\cancel{2^9}$	2^{10}	$2^4 \times 3^2$	$2^4 \times 3^3$	$2^4 \times 3^4$	$2^4 \times 3^5$
2^5	$\cancel{2^7}$	$\cancel{2^8}$	$\cancel{2^9}$	$\cancel{2^{10}}$	2^{11}	$2^5 \times 3^2$	$2^5 \times 3^3$	$2^5 \times 3^4$	$2^5 \times 3^5$
2^6	$\cancel{2^8}$	$\cancel{2^9}$	$\cancel{2^{10}}$	$\cancel{2^{11}}$	2^{12}	$2^6 \times 3^2$	$2^6 \times 3^3$	$2^6 \times 3^4$	$2^6 \times 3^5$
3^2	$\cancel{3^2 \times 2^2}$	$\cancel{3^2 \times 2^3}$	$\cancel{3^2 \times 2^4}$	$\cancel{3^2 \times 2^5}$	$\cancel{3^2 \times 2^6}$	3^4	3^5	3^6	3^7
3^3	$\cancel{3^3 \times 2^2}$	$\cancel{3^3 \times 2^3}$	$\cancel{3^3 \times 2^4}$	$\cancel{3^3 \times 2^5}$	$\cancel{3^3 \times 2^6}$	$\cancel{3^5}$	$\cancel{3^6}$	$\cancel{3^7}$	3^8
3^4	$\cancel{3^4 \times 2^2}$	$\cancel{3^4 \times 2^3}$	$\cancel{3^4 \times 2^4}$	$\cancel{3^4 \times 2^5}$	$\cancel{3^4 \times 2^6}$	$\cancel{3^6}$	$\cancel{3^7}$	$\cancel{3^8}$	3^9
3^5	$\cancel{3^5 \times 2^2}$	$\cancel{3^5 \times 2^3}$	$\cancel{3^5 \times 2^4}$	$\cancel{3^5 \times 2^5}$	$\cancel{3^5 \times 2^6}$	$\cancel{3^7}$	$\cancel{3^8}$	$\cancel{3^9}$	3^{10}

So the distinct products of any two of the given numbers are

(1) 2^4	(2) 2^5	(3) 2^6	(4) 2^7
(5) 2^8	(6) 2^9	(7) 2^{10}	(8) 2^{11}
(9) 2^{12}	(10) 3^4	(11) 3^5	(12) 3^6
(13) 3^7	(14) 3^8	(15) 3^9	(16) 3^{10}
(17) $2^2 \times 3^2$	(18) $2^2 \times 3^3$	(19) $2^2 \times 3^4$	(20) $2^2 \times 3^5$
(21) $2^3 \times 3^2$	(22) $2^3 \times 3^3$	(23) $2^3 \times 3^4$	(24) $2^3 \times 3^5$
(25) $2^4 \times 3^2$	(26) $2^4 \times 3^3$	(27) $2^4 \times 3^4$	(28) $2^4 \times 3^5$
(29) $2^5 \times 3^2$	(30) $2^5 \times 3^3$	(31) $2^5 \times 3^4$	(32) $2^5 \times 3^5$
(33) $2^6 \times 3^2$	(34) $2^6 \times 3^3$	(35) $2^6 \times 3^4$	(36) $2^6 \times 3^5$

(50). The least positive integer divisible by all the single digit numbers is the lcm of 1, 2, 3, 4, 5, 6, 7, 8, and 9 which is $9 \times 8 \times 7 \times 5$ (How?)

$$= 2520.$$

All the multiples of 2, 520 less 10, 000 are also divisible by all the single digit numbers.

i.e., $2520 \times 1, 2520 \times 2, 2520 \times 3$

$(2520 \times 4 \quad 10080 > 10000)$

i.e., 2520, 5040, 7560 are the numbers less than 10,000 and divisible by all single digit numbers.

CHAPTER 11

Sequences

(1). The sequence of natural numbers up to 100 is,
N_{100}: 1, 2, 3, 4, 5, ...98, 99, 100.
(d) The sequence of remainders R_5, dividing each term of N_{100} by 5 is,
$R_5 : \underbrace{1, 2, 3, 4, 0,} \underbrace{1, 2, 3, 4, 0,} \ldots$
The terms t_1, t_2, t_3, t_4, t_5 (*i.e.*, 1, 2, 3, 4, and 0) repeat 20 times.
So the sum of the remainders is
$(1 + 2 + 3 + 4 + 0) \times 20 = 10\times 20 = 200.$
(f) The sequence of remainders, dividing each term by 16 is R_{16} is
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 0, 1, 2, 3, 4, 5 ...0, 1, 2, 3, 4, 5 ...
Here the first 16 terms (*i.e.*, from t_1 to t_{16} which are 1, 2, 3, ...0) repeat itself 6 times up to t_{96}, $t_{97} = 1$, $t_{98}=2$, $t_{99}=3$, $t_{100}=4$.
So here the sum of the remainders is $(1 + 2 + 3 + \ldots + 15 + 0) \times + 1 + 2 + 3 + 4$
$= 120\times 6 + 1 + 2 + 3 + 4 = 720 + 10 = 730.$
(Do the other subdivisions).

(2). Now S is $1^2, 2^2, 3^2, \ldots 100^2$.

Here we shall do, the sequence of remainders dividing by 3 and 8.

$T_3 = 1, 1, 0, 1, 1, 0 \ldots$

Thus 1, 1, 0 repeats 33 times, giving 99 terms, and 100^{th} term is $100^2 = 1$.

Thus the sum of the remainder is

$(1 + 1 + 0) \times 33 + 1 = 67$.

$$T_8 = 1, 4, 1, 0, 1, 4, 1, 0 \ldots$$

Thus the first 4 terms 1, 4, 1, 0 repeats 25 times . So the sum of 100 terms is

$$(1 + 4 + 1 + 0) \times 25 = 150.$$

(3). Here S is $1^3, 2^3, 3^3, \ldots 100^3$.

we do a and c sub division $T_3 = 1, 2, 0, 1, 2, 0, 1, 2, 0, \ldots$

$\therefore$ The sum of there numbers up to 100 terms is

$(1 + 2 + 0) \times 33 + 1 = 99 + 1 = 100$.

$T_4 = 1, 0, 3, 0, 1, 0, 3, 0, \ldots 1, 0, 3, 0$

Thus 1, 0, 3, 0 repeats 25 times . The sum of these remainders up to 100 terms is $4 \times 25 = 100$.

(4). $S = 1, 3, 6, 10, 15, 21, 28, 36, \ldots, \dfrac{100 \times 101}{2} = 5050$,

(c) $R_5 : 1, 3, 1, 0, 0, 1, 3, 1, 0, 0, \ldots$

clearly, the first 5 terms of the sequence of remainders 1, 3, 1, 0, 0 repeats. Thus in 100 terms of the triangular numbers the sum of the remainders is $(1 + 3 + 1 + 0 + 0)20 = 100$.

(e) $R_{10} : 1, 3, 6, 0, 5, 1, 8, 6, 5, 5, 6, 8, 1, 5, 0, 6, 3, 1, 0, 0, 1, 3, 6, 0, 5, 1, 8 \ldots$

It seems(!) that here, after 20 terms, the sequence repeat itself (Verify writing more terms). So, the sum of the remainders is the sum of the 20 terms $\times$ 5

$= 70 \times 5 = 350$.

(5). $t_n = t_{n-1} + t_{n-2}, t_1 = 0, t_2 = 1$
$S = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34.$
Sum to 10 terms $= 0+1+1+\ldots+34 = 88$ (by actual addition).
$11 \times (5 \times 0 \times 8 \times 1) = 11 \times 8 = 88.$

[*Note:* If you take $t_1 = a, t_2 = b, t_n = t_{n-1} + t_{n-2}$, for any $a, b \in N$ it is true].

t_1	t_2	t_3	t_4	t_5
a	b	$a+b$	$a+2b$	$2a+3b$

t_6	t_7	t_8	t_9	t_{10}
$3a+5b$	$5a+8b$	$8a+13b$	$13a+21b$	$21a+34b$

$$\text{Sum to 10 terms} = 55a + 88b.$$
$$= (5 \times t_1 + 8 \times t_2)11.$$

(6). Do it yourself.

(7). Do it yourself.

(8). The first 20 terms of the Fibonacii sequence is F : 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181.

The sequence of squares of these numbers are G: 0, 1, 1, 4, 9, 25, 64, 169, 441, 1156, 3025 ... (complete it).
seq $S = q_n + q_{n+1} = 1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181.$

Clearly S is the sequence where terms are the alternate terms of the initial Fibonacii sequence.
$t_1 + t_2 + t_3 + \ldots 2t_{n-1} = 2t_n = t_{n+1}$
Let us verify by taking n=5
$t_1 + t_2 + t_3 + t_4 + 2 \times t_5$
$= 1 + 2 + 5 + 13 + 68 = 89 = S_6 = F_{12}.$
Verify this for any other value for n. Here is another sequence

where terms are, $t_1, t_1+t_2, t_1+t_2+t_3, \ldots$ of S.
1, 3, 8, 21, 55, ...
All the terms of this sequence is also the alternate terms of the Fibonacii sequence starting from the second term (In fact, starting from 1, these are the alternate terms of Fibonacii sequence.) (You can investigate more properties).

(9).

$$t_1 = 0,\ t_2 = 1,\ t_n = t_{n-1} - t_{n-2},\ \text{for } n \geq 3.$$
$$t_1 = 0,\ t_2 = 1,\ t_3 = 1 - 0 = 1,\ t_4 = 1 - 1 = 0,$$
$$t_5 = 0 - 1 = -1,\ t_6 = -1 - 0 = -1,\ t_7 = -1 - (-1) = 0,$$
$$t_8 = 0 - (-1) = 1,\ t_9 = 1 - 0 = 1,\ t_{10} = 1 - 1 = 0$$

So the sequence is 0, 1, 1, 0, −1, −1, 0, 1, 1, 0, −1, −1, 0 and so on. So the first 6 terms repeat itself
(0, 1, 1, 0, −1, −1) (0, 1, 1, 0, −1, −1) ...
In 100 terms of this sequences we have (0, 1, 1, 0, −1, −1) repeat it self 16 times followed by 4 terms $0+1+1+0$.
$= 0 \times 16 + 1 + 1 = 2.$
So sum to 100 terms of this sequence is 2.

(10). $t_1 = a_1,\ t_2 = a_2,\ t_n = t_{n-1} - t_{n-2}\quad n \geq 3$
So the sequence is

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
$a_1,$	$a_2,$	$(a_2 - a_1),$	$-a_1,$	$-a_2,$	$(a_1 - a_2)$	a_1	a_2

Sum of the first 5 terms $= -a_1 + a_2$ or $a_2 - a_1$
Sum of the first 6 terms $= a_2 - a_1 + a_1 - a_2 = 0.$
$\therefore$ Sum to 100 terms is

$$16 \times 0 + \ \text{the sum of the first four terms}$$
$$= 0 + a_1 + a_2 + a_2 - a_1 - a_1 = 2a_2 - a_1$$

(11). Do it yourself

(12). Do it yourself

(13).

$$t_{123} = \underbrace{123\;123\;123 \cdots 123}_{\text{123 occurs 123 times}}$$

$$t_{244} = \underbrace{244\quad 244\quad \cdots 244}_{\text{244 appears 244 times}}$$

The number of the digit 2 appearing in 244 is $1 \times 244 = 244$ times (*i.e.*, in each block of 244, there is one 2 and there are 244 blocks)

$$\text{Consider}\quad t_{122} = \underbrace{122\quad 122 \cdots 122}_{\text{There are 122 blocks of 122}}$$

The number of 2s appearing in t_{122} is $2 \times 122 = 244$ terms (as there are two 2s in each block of t_{122}). Thus the number of 2s in t_{244} is the same as the number of 2s in t_{122}.

Do the second part.

(14). Sum of the digits $t_{100} = \underbrace{100\quad 100 \cdots 100}_{\text{Blocks of 100 appear 100 times}}$

Thus the sum of the digits is $(1+0+0) \times 100 = 100$

(15). Sequence of number of digits in the terms of t_1, t_2, $\ldots t_n$. 1, 2, 3, 4, 5, 6, 7, 8, 9, 20 (t_{10} =10, 10, 10, …10. There are 10 terms) 22, 24, 26, 28, 30, 32, …100, 102 , …198 (no of digits in t_{99}) 300, (number of digits in 100) 303, 306… Thus the sequence constructed as first 9 of the terms, as single digit numbers from 10^{th} terms up to 99^{th} term, the number of digits form an AP, with first term 20 and common difference 2, then 100^{th} term to 999^{th} term, the number of digits again form are AP with first term 300

and common difference 3. From 1000^{th} term to $9,999^{th}$ term the number of digits of the term it is again an AP with first term 4000, common difference 4. Thus 10^{th} in term is $(n+1)\times 10^n$ and from 10^n term to $(10^{n+1}-1)$ term is an AP with common difference n and (first term $(n+1)\times 10^n$). Answer the second part.

(16). t_{567} is 567 568 569
So sum of the digits in t_{567} is
$5+6+7+5+6+8+5+6+9=57$.
The sequence representing the number of digits t_g is 3, 3, 3, 3, 3, 3, 3, 3, 4, 6, 6 ...6, 7, 9. You can see that $t_9 =$ 9 9 10 having 4 digits but t_{10}=10 10 11, has 6 digits t_{10} to t_{98} we have six digit t_{99}=99 99 100 has 7 digits and t_{100}=100 100 101 has 9 digits, t_{999}=999 999 1000 (10 digits).
If we consider the sequence of number of digits of the initial sequence defined, we have 3, 4, 6, 7, 9, 10, 12, 13, 15, 16. It is easy to see that the terms of this sequence has number of the form 3n, 3n+1 but not 3n+2. The sequence of number which done the form the number of digits of any term is (1, 2), 5, 8, 11, 14, ...(This is an AP from third term on wards). [*Note:* The sequence 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, ...can be written as two sequences with odd ranked terms and even ranked terms as
3 6 9 12 ... (which is an AP: first term 3; C.D-3)
and 4 7 10 · 13 ... (which is also an AP;
first term 4; C.D-3)

(17). The first nine term of the given sequence is 12, 21, 112, 121, 211, 1112, 1121, 1211, 2111, ...(The sequence is constructed using one 2, and one or more 1s.)
The first two terms t_1 and t_2 have 2 digits. The terms t_3, t_4, t_5, have 3 digits. The term t_6, t_7, t_8 and t_9 have 4 digits. Here is a pattern of the terms: the subscripts of the terms t_1, t_3,

$t_6, \ldots$, namely, $(1, 3, 6, \ldots)$, when the number of digits of the term increases from 2 to 3, 3 to 4, 4 to 5,... are the triangular numbers, *i.e.*, numbers of the term $\frac{n \times (n+1)}{2}$. When $n = 1$, we get 1, and when $n = 2$ we get 3. Thus t_1 and t_2 have 2 digits and from t_3, the number of digits is 3 and, there are 3 terms t_3, t_4, t_5 with number of digits 3.

The next term $t_6 = \left(\frac{3 \times 4}{2} = 6 \text{ the } 3^{rd} \text{ triangular number}\right)$ and the term with subscript 3^{rd} triangular number has the digits 4, *i.e.*, 1112.

From the above pattern, the term $t_{\frac{n(n+1)}{2}}$, will have $(n+1)$ digits made up of n, 1s and one 2, and the number of terms having $(n+1)$ digits will be $(n+1)$ terms $t_{\frac{n(n+1)}{2}}, t_{\frac{n(n+1)}{2}+1} \ldots t_{\frac{(n+1)(n+2)}{2}} - 1$.
$28 = \frac{7 \times 8}{2}$, is the 7^{th} triangular number, has 8 digits, with 7 ones and one two and from t_{28} to t_{35} there are 8 digits, with 7 ones and one two, they are
11111112, 11111121, 11111211, 11112111,
11121111, 11211111, 12111111, 21111111.
The number of digits used from t_1 to t_{28} are
2+2, +3+3+3,+4+4+4+4,+5+5+5+5+5,
6+6+6+6+6+6,+7+7+7+7+7+7+7+8
$= (2 \times 2) + (3 \times 3) + (4 \times 4) + (5 \times 5) + (6 \times 6) + (7 \times 7) + 8$
$= 147$ digits.
The number of 2s used up to 28 terms is 28, since each term has exactly one 2.
$\therefore$ no of 1s used up to 28 terms is $147 - 28 = 119$.

(18). Given sequence S is (upto 6 terms)
$S_1 : 12$, $S_2 : 21$, $S_3 : 2112$, $S_4 : 211221$,
$S_5 : 2112212112$, $S_6 : 2112212112211221$
Now T_n: number of 1s in S_n,
U_n: number of 2^s in S_n and

V_n: sum of the digits of S_n.
$T = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,$
$233, 377, 610, 987, 1597, 2584 \ldots$
$U = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89$ (same sequence as T)
$V = 3, 3, 6, 9, 15, 24, 39, 63, 102, 165 \ldots$
Sum of 10 terms of $T = 143 =$ sum to 10 terms of S_2
Sum of 10 terms of $V = 3(143) = 429$
No of 1s used to write 10 terms of S is $1+1+2+3+5+8+13+21+34+55 = 143 =$ No of 2s used to write 10 terms of S.

(19). Do as explained in Problem 17.

(20). Given sequence S is 1, 23, 456, 78910, ... n^{th} term has n natural numbers ending in $\frac{n(n+1)}{2}$. To find the 2005$^{\text{th}}$ term, we should find the last digits of 2005$^{\text{th}}$ term, which is the 2005$^{\text{th}}$ triangular number. 2005$^{\text{th}}$ triangular number is $\frac{2005\times 2006}{2} = 2011015$. The 2005$^{\text{th}}$ term starts with the natural numbers.
$(2011015 - 2004) = 2009011$. So that 2005$^{\text{th}}$ term is 2009011 2009012 2009013 ... 2011014 2011014 2011015 (and there are $2005 \times 7 = 14035$ digits in the term).

(21).

$$t_{10} = 0, \quad t_{11} = 1, \quad t_n = t_{n-1} + t_{n-2}$$
$$\therefore t_{n-2} = (t_n - t_{n-1})$$
$$\therefore \quad t_9 = t_{11-2} = t_{11} - t_{10} = 1 - 0 = 1$$
$$t_8 = t_{10} - t_9 = 0 - 1 = -1$$
$$t_7 = t_9 - t_8 = 1 - (-1) = 2$$
$$t_6 = t_8 - t_7 = -1 - 2 = -3$$
$$t_5 = t_7 - t_6 = 2 - (-3) = 5$$
$$t_4 = t_6 - t_5 = -3 - 5 = -8$$
$$t_3 = t_5 - t_4 = 5 - (-8) = 13$$

$$t_2 = t_4 - t_3 = -8 - 13 = -21$$
$$t_1 = t_3 - t_2 = 13 - (-21) = 34$$

Thus the ten terms are 34, −21, 13, −8, 5, −3, 2, −1, 1, 0.
Extending it to 20 terms, we get,
34, −21, 13, −8, 5, −3, 2, −1, 1,
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55.
Sum of there 20 terms is
$2(34 + 13 + 5 + 2 + 1) + 55 = 3 \times 55 = 165$.

(22). Sum to 100 terms of the given sequence is
$2 + 5 + 11 + 23 + 47 + 19 + 13 + 3 + 7 + 5 + 11 \ldots$
Thus we find that from 10 terms, the eight numbers 5, 11, 23, 47, 19, 13, 3 and 7 repeat. Thus in 100 terms, from 2^{nd} term (5) up to $(8 \times 12) = 96 + 1 = 97^{th}$ term, the above terms repeats 12 times,
98^{th} term is 5,
99^{th} term is 11, and
100^{th} term is 23.
Thus the sum to 100 terms is $2 + 12(5 + 11 + 23 + 47 + 19 + 13 + 7) + 5 + 11 + 23 = 12 \times 128 + 41 = 1577$.

(23). Workout as in Question 21.

(24). Do it yourself.

(25). Do it yourself.

(26). The given sequence of triplets is
(1, 1, 1), (1, 2, 3), (1, 3, 6)...
The general form t_n is $(1, n, \frac{n(n+1)}{2})$.
Since (1, 2005, 2005 × 1003) is of the form $(1, 2005, \frac{2005 \times 2006}{2})$, it is the 2005^{th} term of the sequence of triplet. The terms before it and after it are

t_{2004} is (1, 2004, 2009010),
t_{2006} is (1, 2006, 2013021).

(27). (a)

$$t_{50} - t_{49} \quad \text{of} \quad S = 2^{49} - 2^{48} = 2^{48}$$
$$t_{50} - t_{49} \quad \text{of} \quad T = 3^{49} - 3^{48} = 2 \times 3^{48}$$
$$t_{50} - t_{49} \quad \text{of} \quad U = 5^{49} - 5^{48} = 4 \times 5^{48}$$

(b)

$$2^n - 2^{n-1} = 1 \times 2^{n-1}$$
$$3^n - 3^{n-1} = 2 \times 3^{n-1}$$
$$5^{n+1} - 5^n = 4 \times 5^n \quad \text{and so on.}$$

(i) $2, 6, 18, 54, \ldots 2 \times 3^0, 2 \times 3^1, 2 \times 3^2, 2 \times 3^3 \ldots 2 \times 3^{n-1}$ or $(3^n - 3^{n-1})$

(ii) $5, 5 \times 6, 5 \times 6^2 \ldots 5 \times 6^{n-1} = (6^n - 6^{n-1})$

(28). The product of 2005 terms of 1, $(1 - \frac{1}{2})$, $(1 - \frac{1}{3})$, ... $(1 - \frac{1}{n-1})$, $(1 - \frac{1}{n})$ is $1 \times \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \ldots \frac{2003}{2004} \times \frac{2004}{2005}$
$= \dfrac{1}{2005}$

(29). Given $S_n = \dfrac{n(n+2)}{(n+1)^2}$ of the sequence S, the first 10 terms are,

$$S_1 = \frac{1 \times 3}{2^2}, \quad S_2 = \frac{2 \times 4}{3^2}, \quad S_3 = \frac{3 \times 5}{4^2},$$
$$S_4 = \frac{4 \times 6}{5^2}, \quad S_5 = \frac{5 \times 7}{6^2}, \quad S_6 = \frac{6 \times 8}{7^2},$$
$$S_7 = \frac{7 \times 9}{8^2}, \quad S_8 = \frac{8 \times 10}{9^2}, \quad S_9 = \frac{9 \times 11}{10^2},$$
$$S_{10} = \frac{10 \times 12}{11^2}$$

i.e., $\frac{3}{4}, \frac{8}{9}, \frac{15}{16}, \ldots, \frac{120}{121}$.

$$T = \frac{3}{4}, \quad \frac{2}{3}, \quad \frac{5}{8}, \quad \frac{3}{5}, \quad \frac{7}{12}, \quad \frac{4}{7}, \quad \frac{9}{16}, \quad \frac{5}{9}, \quad \frac{11}{20}, \quad \frac{6}{11},$$

$$U = \frac{3}{4}, \frac{5}{8}, \frac{7}{12}, \frac{9}{16}, \frac{11}{20}, \ldots$$

$$V = \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \frac{6}{11}.$$

The general term t_n of U is $= \frac{2n+1}{4n}$.
The general term of V is $= \frac{n+1}{2n+1}$.
t_n of $U = \frac{2n+1}{4n} > \frac{2n}{4n} = \frac{1}{2}$
and $\dfrac{2n+1}{4n} < \dfrac{2n+2n}{4n} = 1$ $(1 < 2n$ and $n \in N)$.

Thus any term t_n of S_3 is $> \dfrac{1}{2}$ and < 1.

$$t_n \text{ of } V = \frac{n+1}{2n+1} < \frac{n+1+n}{2n+1} = 1$$
$$\text{and } \frac{n+1}{2n+1} > \frac{n+1}{2n+2} = \frac{1}{2}$$

(*Note:* The terms of S on computing gives $\frac{3}{4}, \frac{8}{9}, \frac{15}{16}, \frac{24}{25}, \ldots$) In general if $n-1, n$, and $n+1$, are 3 consecutive natural numbers $(n+1)(n-1) = (n^2-1) < n^2$. Thus the numerator is always 1 less than the denominator. You can find that t_n is $> t_{n-1}$, as

$$S_n = \frac{n \times (n+2)}{(n+1)^2},$$

$$S_{n-1} = \frac{(n-1)(n+1)}{n^2} \quad \text{and}$$

$$\frac{n(n+2)}{(n+1)^2} - \frac{(n^2-1)}{n^2}$$
$$= \frac{(n^2+2n)n^2 - (n^2+2n+1)(n^2-1)}{n^2(n+1)^2}.$$

Numerator is $(n^4+2n^3)-(n^4+2n^3-2n-1) = 2n+1 > 0$ and the denominator is always positive. Thus $S_n > S_{n-1}$ for all n (verify for $n=2$) and the term of T, are decreasing as each term, is multiplied by a number, less than 1. Thus the alternate terms of T, which give rise to sequence U and V are also, decreasing sequences.

(30). a) All the fractions, $\frac{1}{96}, \frac{2}{95}, \frac{3}{94}, \ldots \frac{97}{1}$ are in irreducible form. *i.e.*, the fraction are of the form $\frac{a}{b}$ where a+b = 97.

Now if d is a common divisor of a and b, > 1.

Then $d \mid (a+b) \Rightarrow d \mid 97$. But 97 is a prime number and hence, $d = 1$ or $d = 97$. Thus all fractions of the above sequence are irreducible.

b) Here, the fractions $\frac{a}{b}$ are such that $a+b = 100$. The divisor of 100 are, 1, 2, 4, 5, 10, 20, 25, 40 and 50. Thus when both the numerator and denominator are even numbers, or when both the numerator and denominator are multiples of 5, the fractions are reducible.

Thus $\frac{2}{98}, \frac{4}{96}, \ldots \frac{98}{2}$, are reducible fractions, there are 49 such fractions.

Again $\frac{5}{95}, \frac{10}{90}, \frac{15}{85}, \frac{20}{80}, \ldots \frac{90}{10}, \frac{95}{5}$ are reducible. Hence there are 19 fractions.

However $\frac{10}{90}, \frac{20}{80}, \ldots \frac{90}{10}$, are counted with even numerators these repeated fractions are 9 in number. Thus the number of reducible functions

is $49 + 19 - 9 = 59$.

So, the irreducible fractions are $99 - 59 = 40$.

[The 99 fractions are $\frac{1}{99}, \frac{2}{98}, \ldots \frac{98}{2}, \frac{99}{1}$. Do likewise do the subdivisions c) and d).]

(31). The given sequence $6, 10, 14, 15, 21, 22, 26, \ldots$
$= 2\times 3, 2\times 5, 2\times 7, 3\times 5, 3\times 7, 2\times 11, 2\times 13, \ldots$ So the next

terms can be, 2×17, 3×11, 5×7, 2×19, 3×13, 3×17, 5×11, arranged in ascending orders.

In the given terms, we listed 2×3, 2×5, 2×7, 2×11, 2×13, 3×5, 3×7.

So, we took products of prime numbers with

$$\begin{aligned} 2\times 17 = 34, &\quad 2\times 19 = 38 \\ 3\times 11 = 33, &\quad 3\times 13 = 39, 3\times 17 = 51 \\ 5\times 7 = 35, &\quad 5\times 11 = 55 \end{aligned}$$

These numbers when written in ascending order, gives 33, 34, 35, 38, 39, 51, and 55.

(32). $S_{2n} = 1-2+3-4+5-6+\ldots-2n$. To find $S_{2004}+S_{2005}+S_{2006}$, we shall find the partial series and find a pattern. Let us take $T = 1, -2, 3, -4, 5, -6, \ldots$, where $T_{2n} = -2n$, $T_{2n-1} = 2n-1$.

$$S_1 = 1, \quad = T_1$$
$$S_2 = T_1 + T_2 = 1-2 = -1, \quad \text{also} \quad S_2 = S_1 + T_2,$$

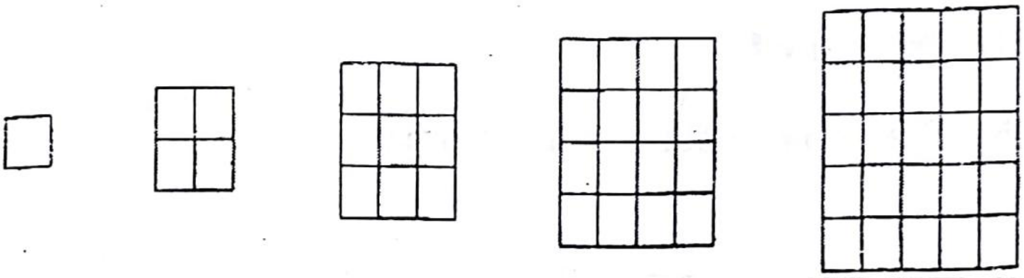
where $T_1, T_2, \ldots$ are the terms of the sequence S

$$\begin{aligned} S_3 &= S_2 + T_3 = -1+3 = 2 \\ S_4 &= S_3 + T_4 = 2-4 = -2 \\ S_5 &= S_4 + T_5 = -2+5 = 3 \\ S_6 &= S_5 + T_6 = 3-6 = -3 \quad \text{and so on.} \end{aligned}$$

We find that $S_{2n} = -n$

$$\begin{aligned} S_{2n+1} &= n+1 \\ S_{2004} &= S_{2\times 1002} = -1002 \\ S_{2005} &= S_{2\times 1002+1} = 1003 \\ S_{2006} &= S_{2\times 1003} = -1003 \end{aligned}$$

$$\text{so} \quad S_{2004}+S_{2005}+S_{2006} = -1002.$$

(33). 

Counting the numbers of squares in the term of the sequence of diagrams, we get,

$$1^2 = 1, \quad 1^2 + 2^2 = 5, \quad 1^2 + 2^2 + 3^2 = 14,$$
$$1^2 + 2^2 + 3^2 + 4^2 = 30, \quad 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

and so on.

Explanation: In the first figure there is exactly one 1 unit square, so $1 = 1^2$

In the second diagram, you can count 4 unit squares and *one* 2×2 square, total numbers of squares is $4 + 1 = 5$.

In the third diagram, there are, 9 unit squares *four* 2×2 squares and *one* 3×3, square totalling $9 + 4 + 1 = 14$. Verify this for the next two figures.

Here, we claim that n^{th} figure will have $n \times n = n^2$ unit squares,
$(n-1) \times (n-1) = (n-1)^2$, 2×2 squares,
$(n-2) \times (n-2) = (n-2)^2$, 3×3 squares,...
One, $n \times n$ square. So the total number of squares that could be counted is

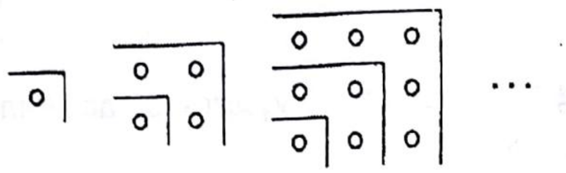
$$1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

[This is the formula for finding the sum of squares of first n natural numbers.] For example, in the 100^{th} figure, total number of squares are

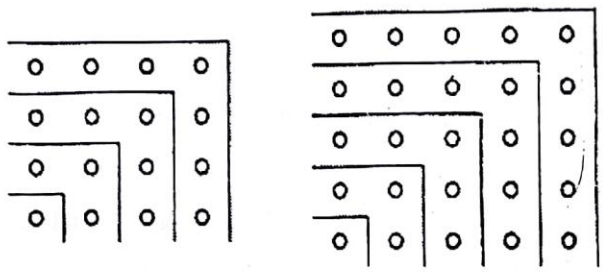
$$\frac{100 \times 101 \times 201}{6} = 50 \times 101 \times 67 = 338350.$$

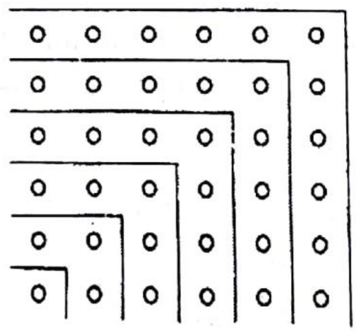
(34). Try yourself.

(35). The following dot diagram is given.



The next 3 diagrams, look like





The sequence of dots in the diagram (as sum of certain numbers) is

$$1 = 1^2, 1 + 3 = 4 = 2^2,$$
$$1 + 3 + 5 = 9 = 3^2,$$
$$1 + 3 + 5 + 7 = 16 = 4^2,$$
$$1 + 3 + 5 + 7 + 9 = 25 = 5^2,$$
$$1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2.$$

Thus the sequence of the sum of odd numbers and the corresponding sequence of their sum is

S_1: (1) $(1+3)$ $(1+3+5)$ $(1+3+5+7)$
$(1+3+5+7+9)$ $(1+3+5+7+9+11)$

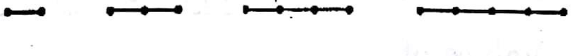
S_2: 1^2 2^2 3^2 4^2 5^2 6^2

But (1), $(1+3)$, $(1+3+5)$, $(1+3+5+7)$, $(1+3+5+7+9)$, $(1+3+5+7+9+11)$, ... represent, sum of first n odd numbers and the n^{th} term is sum of the first n odd numbers. $1+3+5+\ldots(2n-1)$ [*Note:* $(2n-1)$ is the n^{th} odd number. Check!] and this sum is clearly n^2.

[Sum of 1 odd number is 1^2
Sum of first two odd numbers is 2^2
Sum of first three odd numbers is 3^2].
So, the n^{th} figure, will have, n rows of n dots in each row, separated by the slots "$\urcorner$" into first n odd numbers $1+3+5+\ldots(2n-1)$.
So 100^{th} diagram is represented by the sum

$$\underbrace{1+3+5+\ldots+199}_{\text{100 odd numbers}} = 100^2 = 10000.$$

(36).

Sequence of segments				
Number of Segments	1	3	6	10

In the first diagram there is just <u>1</u> unit segment;
in the second diagram, there are <u>2</u> unit segments and 1 segment of length 2 units, so that the total number of segments $1+2=3=\frac{2\times3}{2}$;
In the third diagram, there are 3 unit segments, 2 segments of length 2 units, 1 segment of length 3 unit, and the total number of segments is $1+2+3=6=\frac{3\times4}{2}$;

In the fourth diagram, there are 4 segments of 1 unit length, 3 segments of 2 units length 2 segments of 3 unit length, 1 segment of 4 unit length, thus the total number of segments is $1+2+3+4=10=\frac{4\times5}{2}$.

The tenth segment

Total number of segments is

$$1+2+3+4+5+6+7+8+9+10=55.$$

[*Note:* The number of segments are, the triangular numbers. So the 10^{th} figure will have 11 dots including the end dots and 10 *intervals* of unit segments counting all the segments, gives the 10^{th} triangular number which is $\frac{10\times11}{2}=55$].

(37). Do it yourself.

(38). Do it yourself.

(39). The sequence of diagrams

Terms	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9
Diagrams	△1	$\boxed{2}$	⬠3	△2	$\boxed{3}$	⬠4	△3	$\boxed{4}$	⬠5
Values of diagrams	2	4	27	4	9	64	6	16	125

Terms	t_{10}	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}
Diagrams	△4	$\boxed{5}$	⬠6	△5	$\boxed{6}$	⬠7
values of diagrams	8	25	216	10	36	343

(1) Looking at the numbers inside $\triangle^s$, we find the triangle appear in 1^{st}, 4^{th}, 7^{th}, 10^{th}, ... terms.

Now for a separate sequence of $\triangle$s and numbers in them we get.

$$\begin{array}{ccccc} t_1 & t_4 & t_7 & t_{10} & t_{13} \\ \triangle(1) & \triangle(2) & \triangle(3) & \triangle(4) & \triangle(5) \end{array}$$

Thus in this sequence, the 10^{th} $\triangle$ has 10 in it. and 10^{th} triangle is got by, $1 + 9 \times 3 = 28^{\text{th}}$ term.

$$\begin{aligned} \text{Because } t_1 &= t_{1+0\times3}, \quad \text{first}\triangle, \\ t_4 &= t_{1+1\times3}, \quad \text{second}\triangle, \\ t_7 &= t_{1+2\times3}, \quad \text{third}\triangle, \\ t_{10} &= t_{1+3\times3}. \quad \text{fourth}\triangle. \end{aligned}$$

$$\therefore \text{the } 10^{\text{th}}\triangle = t_{1+9\times3} = t_{28}.$$

So, the 28^{th} term of the sequence has the figure $\triangle(10)$

$\boxed{10}$ follows $\triangle(9)$ and $\widehat{\boxed{10}}$ follows $\boxed{9}$. So $\triangle(9)$ is $t_{1+8\times3} = t_{25}$ And so $\boxed{10}$ is the t_{26} or 26^{th} term.

[*Note:* the terms of the square diagrams are

$t_2 = \boxed{2}$, $t_5 = \boxed{3}$, $t_8 = \boxed{4}$ ...

$t_2 = t_{2+0\times3}$, $t_5 = t_{2+1\times3}$

$t_8 = t_{2+2\times3}$, $t_{11} = t_{2+3\times3} = \boxed{5}$

So $t_{26} = t_{2+8\times3} = \boxed{10}$.]

All the square numbers and even cube numbers, since they are also numbers inside the $\triangle$, will repeat.

Again, if a cube is also a 6^{th} power, it will repeat, as 6^{th} power of a number say a is $a^6 = (a^2)^3$ or $(a^3)^2$

$(a^2)^3 = \widehat{\boxed{a^2}} = \boxed{a^3}$. besides if $a = 2b$ is even, then $\widehat{\boxed{a^2}} = \widehat{\boxed{4b^2}} = \boxed{8b^3} = \triangle(32b^6)$

(40). $S = 1, 12, 123, 1234, \ldots$

$t_n = 1, 2, 3, 4, \ldots (n-1), n$

We shall use a tabular column to count the number of 1s.

Terms From		to	No. of 1s used
t_1	–	t_9	9
t_{10}			2
t_{11}	–	t_{20}	$4+5+6+7+8+9+10+$ $11+12+12=84$
t_{21}	–	t_{30}	$13 \times 10 = 130$
t_{31}	–	t_{40}	$14 \times 10 = 140$
t_{41}	–	t_{50}	$15 \times 10 = 150$
t_{51}	–	t_{60}	$16 \times 10 = 160$
t_{61}	–	t_{70}	$17 \times 10 = 170$
t_{71}	–	t_{80}	$18 \times 10 = 180$
t_{81}	–	t_{90}	$18 \times 10 = 190$
t_{91}	–	t_{99}	$20 \times 09 = 180$
		t_{100}	$21 = 21$

Total number of one's used up to t_1 to t_{100} is 1416.

We shall find the digital root of the terms up to 100 terms.

Terms	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9
Digital roots	1	3	6	1	6	3	1	9	9

Terms	t_{10}	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}	t_{16}	t_{17}	t_{18}
Digital roots	1	3	6	1	6	3	1	9	9

and so on.

From the above values we find the digital roots repeat after 9 terms.

Thus the sum of the digital roots of the terms up to 100 is $11 \times (1+3+6+1+6+3+1+9+9)+1 = 11 \times 39+1$
$= 429+1 = 430$.
Digital root of this sum 430 is 7.

(41). Given the sequence

$$\frac{1}{1}, \quad \underbrace{\frac{2}{1}, \frac{1}{2}}, \quad \underbrace{\frac{3}{1}, \frac{2}{2}, \frac{1}{3}}, \quad \underbrace{\frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}}, \ldots$$

Hint: : Here 8^{th} term lies in the 4^{th} block. We should find in which block, the 50^{th} term lies!

1^{st} block has 1 term,
2^{nd} block has 2 terms,
3^{rd} block has 3 terms.

Thus we group the terms also as follows:

$$t_1 \quad t_2t_3 \quad t_4t_5t_6 \quad t_7t_8t_9t_{10}\ldots$$

Here find that each block ends with a term, the rank of which is a triangular number (*i.e.*, $\frac{n\times(n+1)}{2}$) and each block start with 1 more than a triangular number. More over n^{th} block has n terms. We should find for which n, $\frac{n(n+1)}{2}$ is less than (or equal to) 50.

Clearly n=9 gives $\frac{9\times10}{2} = 45$ and $n = 10$, given $\frac{10\times11}{2} = 55$. Thus, t_{50} lies in 10^{th} block. The terms of 10^{th} block are

$$t_{46},\ t_{47},\ t_{48},\ t_{49},\ t_{50},\ t_{51},\ t_{52},\ t_{53},\ t_{54},\ t_{55}.$$

The fractions in this block are,

$$\frac{10}{1}, \frac{9}{2}, \frac{8}{3}, \frac{7}{4}, \frac{6}{5}, \frac{5}{6}, \frac{4}{7}, \frac{3}{8}, \frac{2}{9}, \frac{1}{10},$$

and the fiftieth term is $\frac{6}{5}$.

(42). In the sequence of fraction, the groups of fraction in which sum of the numerator and denominator is a composite number has some equivalent fraction already appeared in the previous groups of fractions.

(43). The given sequence is

$1, \quad \underbrace{2,3}, \quad \underbrace{4,5,6}, \quad \underbrace{7,8,9,10}, \cdots$

Hint: n^{th} group has n natural numbers.
Name the terms as $t_1\ t_2\ t_3 \cdots$ and follow the method described in Question 41. $\frac{13\times14}{2} < 100 < \frac{14\times15}{2}$.

(44). In Question 43, you may observe that the group of numbers have a general form. The n^{th} group has n numbers,

$$\frac{n(n-1)+2}{2}, \frac{n(n-1)+4}{2}, \frac{n(n-1)+6}{2}, \cdots$$
$$\frac{n(n-1)+2n}{2}\left(=\frac{n(n+1)}{2}\right)$$

Thus, the last number of the tenth group is $\frac{10\times11}{2} = 55$, the first number is $\frac{10\times9+2}{2} = 46$
The sequence required upto 10 terms is

$$S: 1, \quad 5, \quad 15, \quad 34,$$
$$\left[1, \left(\tfrac{3\times4}{2}-1\right), \left(\tfrac{6\times7}{2}-\tfrac{3\times4}{2}\right), \left(\tfrac{10\times11}{2}-\tfrac{6\times7}{2}\right)\right.$$
$$65, \quad \ldots \quad 505$$
$$\left.\left(\tfrac{15\times16}{2}-\tfrac{10\times11}{2}\right), \quad \ldots \quad \tfrac{55\times56}{2}-\tfrac{45\times46}{2}\right]$$

Sum of these numbers to $1, 2, 3, \ldots 10$ terms are
$1, 6, 21, 55, 120, \ldots 1540$.
Again you may find that these sum are

$$\frac{1\times2}{2}, \frac{3\times4}{2}, \frac{6\times7}{2}, \frac{10\times11}{2}, \frac{15\times16}{2}, \cdots \frac{55\times56}{2}.$$

Note: Every term of the sequence of the sum S. is the difference between two triangular numbers.

The sum of the terms of S is again a triangular number. The sum to 10 terms being 55^{th} triangular number; 55 itself is the 10^{th} triangular number.

Thus sum to n terms of S is given by finding the n^{th}triangular number $\left(\text{which is} \frac{n(n+1)}{2}\right)$ and the sum is

$$\frac{\frac{n(n+1)}{2} \times \frac{n(n+1)+2}{2}}{2} = \frac{n^2(n+1)^2 + 2n(n+1)}{8}$$

(45). For you to do.

(46). For you to do.

Chapter 12

Arithmetic of Remainders (Modulo Arithmetic)

(1). *Hint:* 10, 19, ...91 leave remainder 1 on dividing by 9. *Note:* The digital root in each case is 1.
Difference of any 2 numbers in the above list is always, a multiple of 9. and hence leave a remainder '0' on dividing (this difference) by 9 other subdivision to be done by you.

(2). Do it by yourself.

(3). Do the sums.

(4). Do the sums.

(5). Do the sums.

(6). Do the sums.

(7). Do the sums.

(8). (a)

$$32^{12} \equiv x \pmod{13}$$

$$32^1 \equiv 6 \pmod{13} (32 - 6 = 26, 13 \mid 26)$$
$$32^2 \equiv 6^2 = 36 \equiv 10 \pmod{13}$$
$$(32^2)^2 = 32^4 \equiv 10^2 = 100 \equiv 9 \pmod{13}$$
$$32^8 \equiv 9^2 = 81 \equiv 3 \pmod{13}$$
$$32^{12} = 32^8 \times 32^4 \equiv 3 \times 9 = 27 \equiv 1 \pmod{13}$$

Workout the other subdivisions.

(9). (c)

$$4324 \equiv 4 \pmod{10}$$
$$4324^2 \equiv 6 \pmod{10}$$
$$4324^4 \equiv 6 \pmod{10}$$
$$4324^8 \equiv 6 \pmod{10}$$

Thus it can be observed that any even power of 4324 is $\equiv 6 \pmod{10}$.
$\therefore 4324^{500} \equiv 6 \mod 10$
Thus the unit digit of 4324^{500} is 6.

(10). (a) $n \equiv 1 \pmod 4$ means $(n-1)$ is divisible by 4.
Thus the two digit numbers divisible by 4 are 12, 16, ...96.
Thus there are 22, two digit numbers divisible by 4.
$\therefore n = 13, 17, 21, \ldots 97$.

(b) Do subdivision (b) and (c), and then find the common numbers appearing in the solution of (a), (b) and (c).

(c) $n \equiv 1 \pmod{16} \Rightarrow (n-1)$ is divisible by 16 and each ns are 16, 32, 48, 64, 80, 96 and the corresponding values of $n-1$ = 17, 33, 49, 65, 81, 97. Thus $17 \equiv 1 \pmod{16}$ etc.
Note: all numbers $16k+1$ satisfy the condition of the problem.

(11). $(n-1)$ is divisible by 5, 7 and 3. So, $(n-1)$ should be common multiple of 5, 7 and 3. *i.e.*, multiple of 105. The least such

number itself is 105 a three digit number.
Thus $n = 106$, a three digit number.
So, there exists no two digit number, which leaves a remainder 1 on division by each of 3, 5 and 7.

(12). a) $(n-1)$ is divisible by 5 and also $(n-3)$ is divisible by 4. *1st method:* Two digit numbers divisible by 5 are 10, 15, 20, 25, ...95. Thus $n = 11, 16, 21, 26, \ldots 96$. Of the numbers 11, 16, 21, 26, ...96. The numbers leaving a remainder 3, on division by 4 are

$$11, 31, 51, 71, 91. \tag{a}$$

Note: that the difference between any two Consecutive numbers of a is $20 = 4 \times 5$.

Another method

$$n \equiv 1 \mod 5$$
$$\Rightarrow (n-1) \text{ is a multiple of 5 say } 5k$$
$$\Rightarrow n = 5k + 1$$

Similarly $n - 3$ is divisible by 4, making $n = 4m + 3$

$$\text{Thus } 4m + 3 = 5k + 1$$
$$\Rightarrow 4m = 5k - 2$$
$$\Rightarrow 4m = 4k + (k-2)$$
$$\Rightarrow m = k + \frac{k-2}{4}$$

and $(k-2)$ should be divisible by 4. So, $k = 2, 6, 10, \ldots$ makes $(k-2)$ divisible by 4. So, there is a table of values of k and the corresponding values of m and the values of n.

k	2	6	10	14	18	22
m	2	7	12	17	22	27
n	11	31	51	71	91	111 ...

of the values of n, 11 to 91 are two digit number. You can get infinitely many values for n, if there is no restriction about the number of digits.

$$\begin{aligned} 2, 7, 12, \cdots &\quad \text{are AP with C.D. 5 and} \\ 2, 6, 10, \cdots &\quad \text{are AP with C.D. 4 and} \\ n = 11, 31, 51, \cdots &\quad \text{are AP with C.D. } 4 \times 5 = 20 \end{aligned}$$

b) $n \equiv m \pmod{20}$, gives $m = 11$, for all values of n of the list. $n \equiv m \pmod{40}$, gives $m = 11$, for, $n = 11, 51, 91$ etc. and $m = 31$ for n=31, 71, $\underbrace{101}_{\text{3 digit numbers}}$ etc.

(13). $100 - 1$ is divisible by d. i.e., 99 is divisible by d. i.e., d is a divisor of 99 (we take $d > 1$ as 1 divides all numbers). The non unit divisor of $99 = 3^2 \times 11$ are 3, 9, 11, 33, and 99. Thus the possible values of d are 3, 9, 11, 33 and 99.

(14).

$$\begin{aligned} a &\equiv 18 \pmod 7 \\ b &\equiv 42 \pmod 7 \\ (a+b) &\equiv 60 \pmod 7 \\ &\equiv 4 \pmod 7 \\ \text{or} \quad a &\equiv 4 \pmod 7 \\ b &\equiv 0 \pmod 7 \\ \therefore a + b &\equiv 4 \pmod 7 \end{aligned}$$

Thus the smallest value of c is 4.

(15). Here the table is partially filled. Fill up the blank cells.

$+_7$	0	1	2	3	4	5	6
0	0						6
1		2				6	0
2			4		6		1
3				6			2
4			6		1		3
5		6				3	4
6	6						5

(16). Fill up the other cells

$\times_7$	1	2	3	4	5	6
1	1			4		6
2		4		1	3	
3			2	5		
4			5	2		
5		3		6	4	
6	6			3		1

(17). a)

$$x + 5 \equiv 3 \pmod 7$$
$$\therefore\ x \equiv 5 \pmod 7$$

b)

$$x + 4 \equiv 1 \pmod 7$$
$$x \equiv 2 \pmod 7$$

c)

$$5 \times x + 3 \equiv 6 \pmod 7$$
$$5 \times x \equiv 3 \pmod 7$$
$$x \equiv 2 \pmod 7$$

(18).

$$\begin{aligned} 4 \times_7 (6 +_7 5) &\equiv 4 \times_7 (4) \pmod 7 \\ &\equiv 2 \pmod 7 \\ &\equiv (4X_7 6) +_7 (4X_7 6) \pmod 7 \\ &\equiv 3 +_7 6 \pmod 7 \\ &\equiv 2 \mod 7 \end{aligned}$$

(19). Do it yourself

(20). Do it yourself

(21). To show that:
$127^{2005} + 721^{2005} + 217^{2005} + 271^{2005}$ is divisible by 7. We should find the modulo $7(127^{2005}) + 721^{2005} + 217^{2005} + 271^{2005}$

$$\begin{array}{llll} \text{(i)} & 127 & \equiv 1 & \pmod 7 \\ & 127^{2005} & \equiv 1^{2005} & \pmod 7 \\ & & \equiv 1 & \pmod 7 \\ \text{(ii)} & (721)^{2005} & \equiv 0 & \pmod 7 \\ \text{(iii)} & (217)^{2005} & \equiv 0 & \pmod 7 \quad \text{Why?} \\ \text{(iv)} & (271) & \equiv 5 & \pmod 7 \quad \text{Why?} \\ & 271^2 & \equiv 4 & \pmod 7 \\ & 271^3 & \equiv 20 & \pmod 7 \\ & 271^4 & \equiv 2 & \pmod 7 \\ & 271^5 & \equiv 3 & \pmod 7 \\ & 271^6 & \equiv 1 & \pmod 7 \end{array}$$

Thus every 6^{th}power of 271 is 1 (mod 7) 2005=($6\times 334 + 1$)

$$\begin{aligned} \therefore (271)^{2005} &= \left((271)^6\right)^{334} + 271^1 \\ &= (271^6)^{334} + (271)^1 \pmod 7 \\ &= 1 + 5 \pmod 7 \end{aligned}$$

$$= 6 \pmod 7$$

$$\therefore 127^{2005} + 721^{2005} + 217^{2005} + 271^{2005}$$
$$\equiv (1 + 0 + 0 + 6) \bmod 7$$
$$\equiv 0 \pmod 7$$

$\therefore$ The number
$127^{2005} + 721^{2005} + 217^{2005} + 271^{2005}$ is divisible by 7.

(22). Do it yourself.

(23).

$$\begin{aligned}
2005^1 &\equiv 5 \pmod{100} \\
(2005)^2 &\equiv 25 \pmod{100} \quad \text{(Why?)} \\
(2005)^4 &\equiv 25 \pmod{100} \\
&\vdots \\
&\vdots \\
(2005)^{2006} &\equiv 25 \pmod{100} \quad \text{(Explain)} \\
2006 &\equiv 06 \pmod{100} \\
2006^2 &\equiv 36 \pmod{100} \\
2006^3 &\equiv 16 \pmod{100} \\
2006^4 &\equiv 96 \pmod{100} \\
2006^5 &\equiv 76 \pmod{100} \\
2006^6 &\equiv 56 \pmod{100} \\
2006^7 &\equiv 36 \pmod{100} \\
\vdots \quad & \quad \vdots
\end{aligned}$$

Now, you can verify, that 2006^2, 2006^3, 2006^4, 2006^5, 2006^6 are $\equiv 36, 16, 96, 76, 56 \pmod{100}$ and this cycle repeats for (2006^7,

$2006^8, \ldots 2006^{11})$ 2006^{12}, 2006^{13}, $\ldots (2006^{17}, 2006^{18} \ldots)$
$(2006) \equiv (2006)^x \equiv 36 \pmod{100}$ for $x = 2, 7, 12, 17, \ldots$ and so on. 2, 7, 12, 17... is an AP. These numbers are of the form $5k+2, k = 0, 1, 2 \ldots 2005 \equiv 2002 + 3 = (5 \times 400 + 2) + 3$

$$\text{So} \quad 2006^{2005} = (2006)^{(5\times400+2)} \times 2006^3$$
$$2006^{2002} = (2006)^{(5\times400+2)} \equiv 36 \pmod{100}$$
$$2006^3 \equiv 16 \pmod{100}$$
$$\therefore (2006)^{2005} \equiv 36 \times 16 \pmod{100}$$
$$= 76 \pmod{100}$$
$$(2005)^{2006} + (2006)^{2005} \equiv (25 + 76) \pmod{100}$$
$$= 101 \pmod{100}$$
$$= 01 \pmod{100}$$

(24). Do it yourself.

(25). *Hint:* $117 = 9 \times 13$. $\therefore$ Show that $23984^{15} + 4119926^{25} + 359776^{35} + 719551^{45}$ is divisible by 9 as well as by 13 using modulo 9 and modulo 13.

(26). Any odd square number is $1 \pmod 4$ and any even square number is $0 \pmod 4$. Let the two odd numbers be $2n+1$ and $2m+1$.

$$(2n+1)^2 = 4n^2 + 4n + 1 \equiv 1 \pmod 4 \qquad \text{(A)}$$
$$(2m+1)^2 = 4m^2 + 4m + 1 \equiv 1 \pmod 4 \qquad \text{(B)}$$

Adding (A) and (B) we get $(2n+1)^2 + (2m+1)^2 \equiv 2 \pmod 4$. Thus $(2n+1)^2 + (2m+1)^2$ cannot be a square number. Similarly $(2n+1)^2 + (2m+1)^2 + (2p+1)^2 \equiv 3 \pmod 4$ and hence sum of the squares of any three odd numbers cannot be a square number.

CHAPTER 13

Pigeon Hole Principle

Pigeon Hole Principle: If there are n pigeon holes and m pigeons, where $m > n$, then at least one hole will have at least 2 pigeons.

If there are say 5 holes and six pigeons, then after each hole is occupied by one pigeon, there is not a hole for one pigeon to occupy. So this pigeon, should choose one of the five holes to go into, and that hole will have 2 pigeons. *Note:* There is no condition that all the pigeons should not enter into the same hole!

(1). A teacher asked a group of ten students, each to write a non zero single digit number.

Then, even if no two students write the same number, then only nine of them can write numbers from 1 to 9, *i.e.*, each writing a number, these nine numbers from 1 to 9 would have been written by just nine students. So the 10^{th} student should write one of 1 to 9 numbers, which has been already written. Thus at least two students would have written the same number.

Note: If some of the numbers, are not written by any students, then, more than two students would have written the same number.

Here, the numbers 1 to 9 are the nine holes and the 10 students are the ten pigeons.

(2). In a year there are twelve months—Jan to Dec;
these 12 months are the holes. the 13 boys celebrating birth days are the pigeons.
Since pigeons are more than the holes the result follows.
That is at least two of the boys would have their birth days on the same month.

(3). Do it your self

(4). The four pairs of socks (of the same size and colour) would have 4 socks for the right leg and four for the left leg.
If you choose 3 socks; surely two of three, will have one left leg socks and 1 right leg sock.
If you choose 2, both of them may turn out to be left (or right) so, you can not use it.

(5). There are 10 yellow gloves (5 of them for left arm and 5 for the right arm) and 10 pink gloves. (There are totally 10 gloves for left arm and 10 for right arm.) Here if you choose 3 gloves, you may have two yellow gloves both to be used for right and one pink gloves. If you choose 10 gloves again, it is possible that 5 yellow right arm gloves and 5 pink right arm (or left arm) gloves could have been drawn, so that no pair could be used.
If you choose 11 gloves then, even if you get 5 yello right arm gloves, and 5 pink left arm gloves (no two of which can be used), the 11th glove may either be pink and right arm or yellow and left arm, so that this 11th gloves will either match with pink gloves drawn or yellow gloves drawn.
Thus, the least number of gloves to be drawn, so that a pair of them may be of use is 11.

(6). The number of blue balls 10 number of red balls 7.

a) To draw at least one red ball, the least number of balls to be drawn is 11 (if 10 balls are drawn it is possible that all of them may be blue balls)

b) Drawing 8 balls guarantees the presence of 1 blue ball.

c) Drawing 3 balls guarantees in obtaining at least 2 balls of the same colour.

d) Drawing 5 balls guarantees in obtaining 3 balls of the same colour (4 balls may give rise to 2 red and 2 blue balls.

(7). Do it your self

(8). Do it your self

(9). a) The dates 1, 8, 15, 22, 29 leave remainder 1 on division by 7.

b) The dates 2, 9, 16, 23, 30 leave remainder 2 on division by 7.

c) The dates 3, 10, 17, 24, 31 leave remainder 3 on division by 7.

d) The dates 4, 11, 18, 25 (there are just 4 different numbers) gives remainder 4 on division by 7.

e) The dates 5, 12, 19, 26 give remainder 5.

f) 6, 13, 20, 27 give remainder 6 and

g) 7, 14, 21, 28 give remainder on division by 7. Thus the remainders 1, 2 and 3 repeat 5 times (an dividing the dates by 7). All the months having 31 days have this property, *i.e.*, Jan, March, May, July, Aug, Oct, Dec have this property.

(10). a) Here are 10 numbers so, we can pair them as (5, 85), (15, 75), (25, 65), (35, 55). So that the numbers of each pair add up to 90.

If we choose one number from each pair. We get 4 numbers and, we can choose 45 and 85 as two of the numbers. Thus we have 6 numbers, no two of these number add to 90. However, if one more number to be selected, this could be done from, the numbers left out of the pairs. Thus, choosing the 7^{th} number, result in choosing two numbers from the same pair, these two numbers will add up to 90. (Here pigeon holes are six, numbers to be choosen (*i.e.*, 7) is the pigeons).

b) *Hint:* now pair the numbers as (5, 95), (15, 85), (25, 75), (35, 65), (45, 55). You should choose 6 numbers, from out of 5 pairs, choosing 6^{th} number give the result.

(11). Let us assume that in the 7 days of the week, no three twin babies were born on the same day, triplet and quadruplets were not born on the same day, neither on the day, when two twin babies were born.

Without loss of generality we use the following table of 7 days, for the day of birth of twins, triplets and quadruplets. The total number of babies born is 57.

Twin, quadruplets, Triplets total-up to $12 \times 2 + 3 + 4 = 31$

The babies born as a single baby is $57 - 31 = 26$. Tabulating this, we get

	Sun	Mon	Tue	Wed	Thurs	Fri	Sat
	1^{st} Twin	2^{nd} Twin	3^{rd} Twin	4^{th} Twin	5^{th} Twin	6^{th} Twin	7^{th} Twin
	(2)	(2)	(2)	(2)	(2)	(2)	(2)
	8^{th} Twin	9^{th} Twin	10^{th} Twin	11^{th} Twin	12^{th} Twin	Triplet	Quad-ruplet
	(2)	(2)	(2)	(2)	(2)	(3)	(4)
Total*	4	4	4	4	4	5	6

* without counting single babies

Distributing the single babies (26) equally to all the 7 days, each day there should be 4 babies, for 5 of the days and for the remaining two of the days, 3 babies each.
In the above table if we include the 26 babies as explained above we get,

	Sun	Mon	Tue	Wed	Thurs	Fri	Sat
	4	4	4	4	4	5	6
	4	4	4	4	4	3	3
Total	8	8	8	8	8	8	9

Thus, at least in one day of the week (in the tabular column above, it is a Saturday) 9 babies were born. [*Note:* ; For any other record of the babies born, one day may even have more than 9 and another day less than 8]

(12). Even if, from each pockets, the pins drawn are non-defective, then such maximum number of pins would be

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}
Defective pairs	4	3	1	2	3	4	3	3	2	2
Pins without defect	6	7	9	8	7	6	7	7	8	8

only $(6+7+9+8+7+6+7+7+8+8) = 73$. Total pins drawn is 80. So, there should at least be 7 defective pins in the choosing 80 pins. there are just 73 nondefective pins, so the other 7s are to be taken from among the pins left in the pockets which contain only defective pins. Choosing at random, the defective pins may be 7 or more than 7.

(13). (Refer to Problem 10).
These 100 numbers from 1 to 100, on pairing, we get.
(1, 100), (2, 99),... (50,51), gives fifty pairs. If 51 numbers are to be choosen, choosing 50. numbers first one from each pair, we should choose the 51st number, from among the numbers left out in the pairs. Thus, in the selection of 51, numbers 2 of the numbers are selected from the same pair which add up to 101. Thus, you cannot choose 51 numbers without two numbers adding up to 101. (But 50 number could be chosen, so that no two numbers add up to 101.)

(14). *Hint:* The pairs are (3, 100),(4, 99),...... (49, 54),(50, 53), (51, 52). the total number of pairs are 49. The total numbers used to get the pairs are 98. The numbers not used in the pairs are 1 and 2. Thus numbers can be chosen as, 1, 2, and 49 other numbers, one from each of 49 pairs. So the 52nd number should be selected from among the numbers not chosen from the 49 pairs. Thus, 52nd number, and another number chosen from 49 pairs add up to 103. Thus, it is not possible to choose 52 numbers without two numbers adding up to 103. *i.e.*, there is always (at least) one pair of numbers, whose sum is 103.

(15). Now you can group the numbers from 1 to 100, according to the remainders, got on dividing by 11. Denote the numbers leaving remainder 0, 1, 2, 3, ... 10 as $R_0, R_1, R_2, R_3 \ldots R_{10}$.

						Total No.
R_1:	1,	12,	23,	......,	100	10
R_2:	2,	13,	24,	......,	90	9
R_3:	3,	14,	25,	......,	91	9
R_4:	4,	15,	26,	......,	92	9
R_5:	5,	16,	27,	......,	93	9
R_6:	6,	17,	28,	......,	94	9
R_7:	7,	18,	29,	......,	95	9

R_8:	8,	19,	30,	......,	96	9
R_9:	9,	20,	31,	......,	97	9
R_{10}:	10,	21,	32,	......,	98	9
R_0:	11,	22,	33,	......,	99	9

Now you can choose all number from R_1, R_2, R_3, R_4, R_5 and one number (divisible by 11) from R_0.
This gives a total number of numbers as $10 + 9 \times 4 + 1 = 47$ numbers. Here in these 47 numbers no two numbers give a sum divisible by 11.
But you have to choose one more number; This could be choose from among the numbers R_6, R_7, R_8, R_9, R_{10}, R_0 (just one number is included, in the previous collection of 47 numbers, so there are 8 more numbers in R_0).
If the 48^{th} number is choosen from R_6 this number added to the already choosen number from R_5 is divisible by 11 and similarly if the number is choosen from the other group also, there is already a number more than in the 47 numbers choosen so that this 48^{th} number added to that number gives a sum divisible by 11.
Note: Instead of taking R_1 R_2 R_3 R_4 R_5 and one number from R_0, you can take 47, numbers from R_1 R_2 R_3 and R_6 R_7 and one numbers from R_0.

(16). Consider the 2006 numbers, made up of 1s as follows.
1, 11, 111, ...1111...111, 111...111, ...Since there are 2006 numbers (whose digits are units) there are 2006 remainders when each is divided by 2005.
But, when numbers are divided by 2005, the possible remainders are 0 to 2004. *i.e.*, only 2005. Since there are 2006 remainders two of the numbers should leave the same remainder on dividing by 2005. Say R_m and R_n with $m > n$. (R_m having m ones and

R_n having n ones).

$$\therefore R_m - R_n = \underbrace{(1,111,\ldots 111)}_{m \text{ ones}} - \underbrace{(111\ldots 111)}_{n \text{ ones}}$$

$$= \underbrace{111\ldots}_{(m-n)\text{ ones}}\ \underbrace{000\ldots 0}_{n \text{ zeros}}$$

But this number $\underbrace{111\ldots 1111}_{m-n \text{ ones}}\ \underbrace{0000\ldots 0}_{n \text{ zeros}}$ is divisible by 2005,

$$R_m = 2005k - r, \quad R_n = 2005l - r$$

$$R_m - R_n = (2005k - r) - (2005l - r) = 2005k - 2005l$$

$$= 2005(k - l) \quad \text{is a multiple of 2005.}$$

(17). Do as in Problem 16, replacing 1 by a.

(18). If none of the numbers a_1, a_2, a_3, a_4, a_5 is divisible by 5, these five numbers may leave any of the four remainders 1, 2, 3 and 4. Now consider the series:

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, a_1 + a_2 + a_3 + a_4 + a_5$$

It is possible that any one of the numbers may be divisible by 5. If not, two of the numbers, will leave the same remainder out of 1, 2, 3, 4. Difference of these two numbers, leaving the same remainder is again sum of two or several of a_1, a_2, a_3, a_4, a_5.
Example: if $a_1 + a_2 + a_3 + a_4$ and $(a_1 + a_2)$ leave the same remainder, then

$$(a_1 + a_2 + a_3 + a_4) - (a_1 + a_2) = a_3 + a_4$$

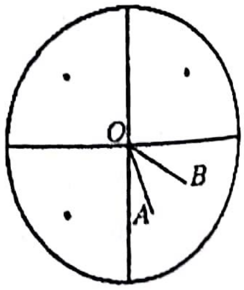
is divisible by 5.
Note: (1) a_1 and $a_1 + a_2$ do not leave the same remainder as $a_1 + a_2 + -a_1 = a_2$ is not divisible by 5 by our assumption.
Note: (2). From out of these 5 numbers, $a_1, a_1 + a_2, a_1 + a_2 +$

$a_3, a_1 + a_2 + a_3 + a_4, a_1 + a_2 + a_3 + a_4 + a_5$, you can get 10 pairs of numbers:

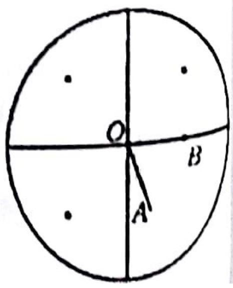
(1) $(a_1, a_1 + a_2)$,
(2) $(a_1, a_1 + a_2 + a_3)$,
(3) $(a_1, a_1 + a_2 + a_3 + a_4)$,
(4) $(a_1, a_1 + a_2 + a_3 + a_4 + a_5)$,
(5) $(a_1 + a_2, a_1 + a_2 + a_3)$,
(6) $(a_1 + a_2, a_1 + a_2 + a_3)$,
(7) $(a_1 + a_2, a_1 + a_2 + a_3 + a_4)$,
(8) $(a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4)$,
(9) $(a_1 + a_2 + a_3 + a_4 + a_5)$,
(10) $(a_1 + a_2 + a_3 + a_4, a_1 + a_2 + a_3 + a_4 + a_5)$

If none of a_1, a_2, a_3, a_4, a_5 is divisible by 5 then the 4 pairs (1, 5, 8 and 10) may not leave the same remainder. Considering the other 6 pairs, there should be at least one pair such that they leave the same remainder, and the the difference of these numbers is divisible by 5, and the difference is sum of all of a_1, a_2, a_3, a_4, a_5 or of some of the numbers from a_1 to a_5.

(19). The circle can be divided into 4 quadrants. Since 5 points are to be placed inside the circle, 4 of the points may the placed, one in each of the four quadrants. So, the fifth point should have a quadrant with a point already in it say A and B. Now $\angle AOB$ is less than 90° (Why?) and hence AOB is an acute angle.



Note: Even if one point lies on a diameter (refer figure here), then $\angle AOB$, is an acute angle.



(20). Pairing the given numbers so that the product of the numbers into the pairs is 36, $(1, 36)$, $(2, 18)$, $(3, 12)$, $(4, 9)$. We are left with the numbers six. Choosing one number from each pair and 6, we get first 5 numbers to get the sixth number , we should take one of the four numbers left out in the pairs. Thus, we have chosen, two numbers from the same pair, The product of these two numbers is 36.

(21). Let us name the students as s_1, s_2, s_3, s_4 and s_5. (We leave out the case that every one is his own friend!) If s_1 has 1 friend s_2 has two friends, s_3 has 3 friends and s_4 has four, friends s_5 should have 1 or 2 or 3 or 4 friends (since we leave out the case, that everyone is his own friend!) Thus if he has 1 friend , he has as many friend as s_1, 2 friends, he has as many as friends as s_2 and so on.
Thus, in the group there are two students, (s_5 and another) who have identical number of friends.

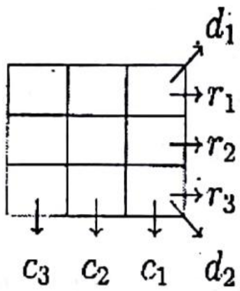
(22). Consider the 2006, powers of $3, 3^{n_1}, 3^{n_2}, 3^{n_3} \ldots 3^{n_{2005}}, 3^{n_{2006}}$ $ni \neq 0$ for any i. Since there are 2006 numbers are divided by 2005, there are utmost 2005, different remainders 0, 1, 2, …2004. Thus, two of the 2006 numbers should have the same remainder on dividing by 2005, say, 3^m and 3^k where m and k are two distinct numbers among n_1, n_2, n_3 …, and $3^m - 3^k$ is a multiple of 2005.

(23). The three numbers 1, 2, 3, can give only 7 distinct sums. All possible sums are

$$1 + 1 + 1 = 3$$
$$1 + 1 + 2 = 4$$
$$1 + 1 + 3 = \boxed{5}$$

$$1+2+3=6 \qquad 2+2+2=6$$
$$1+2+2=\boxed{5} \qquad 2+2+3=7$$
$$1+3+3=7 \qquad 3+3+3=9$$
$$3+3+2=8$$

of these 10 sums, 3, 4, 9 and 8 appears exactly 1, and each of 5, 6 and 7 appears twice. Thus there are exactly 7 different sums, 3, 4, 9, 8, 5, 6,and 7 But, when these 3 numbers 1, 2, 3, are arranged we get, 3 sums adding along the rows, 3 sums adding along the columns, and 2 sums adding along the diagonals. So there are 8 sums and hence at least one of the sum should repeat.



Note: If one of the sums is to be 9 (or 3) you should have one column or one row or one of diagonals to have three 3ˢ. Thus in any case, 3 occurs at least once in each row (or each column), so that you cannot get another sum as 3 or 4, using 9 leaves you a choice of 9, 8, 5, 6 and 7 to be the 8 sums, so at least three different series repeat twice. In the square cells given below, you find that 6 appears thrice 5 appears twice.

Can you construct square cells, so that no sum repeats thrice?

6				
	3	3	3	9
	1	2	2	5
	1	2	1	4
6	5	7	6	

(24). If each boy has gathered different number of shells, say 1 to 10, the total shells collected becomes $\frac{10\times11}{2} = 55 > 40$. But only 40 shells are collected. If 8 of them collected shells numbering 1 to 8, the total number of shells become 36. So, 4 more shells are to be collected by the say, 9^{th} and 10^{th} boys, which could be 1+3, 2+2 or 3+1, in which case, the first boy and 9^{th} boy (or 3^{rd} boy and 9^{th} boy) collected the same number of shells. Thus at least two of their collected the same number of shells.

Note: If one of the ten boys did not collect any shell *i.e.*, collected 0 shell, then, the number of shells collected can be taken as 0, 1, 3, 4, 5, 6, 6, 7, 8 and now two of these boys collected 6 shells each. This is the same as taking 9 boys collected together 40 shells. (each collected at least 1 shell) then there should be two boys who should have collected the same number of shells. Investigate more on this sum.

(25). The product of all the six numbers is $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$. If all the 3 groups have the same product say p, then $p \times p \times p$ (there are 3 groups) $p^3 = 720$ and 720 is not a cube number and hence, the product of numbers in all the 3 groups can not be equal *i.e.*, at least two of them should have different products. But of these two products one product should exceed the other.

[Example: $720 = 6 \times 6 \times 20$ can be split up into

$$(2 \times 3)\times(1 \times 6)\times(4 \times 5).$$

Here 4×5 exceeds 1×6 (also 2×3).
2×5, (1×3), (4×6) is another example.]

Chapter 14

Algebra

(1).

$$a^2 - b^2 = 17$$

$$(a+b)(a-b) = 1 \times 17 \quad \text{or} \quad 17 \times 1 \quad \text{or}$$

$$= -1 \times -17 \quad \text{or} \quad -17 \times -1$$

(i)

$$a + b = 1$$

$$a - b = 17$$

gives $\quad a = 9, b = -8$

(ii)

$$a + b = 17$$

$$a - b = 1$$

gives $\quad a = 9, b = 8$

(iii)

$$a + b = -1$$

$$a - b = -17$$

gives $\quad a = -9, b = 8$

(iv)

$$a + b = -1$$

$$a - b = -17$$

gives $\quad a = -9, b = -8$

Thus there are 4 sets of values for (a, b); (9, 8), (-9, 8), (9, -8), (-9, -8)

Note: Students of class IV and V can take natural number solutions which is (9, 8) according to (ii).

(2). *Hint:* $12 = 1 \times 12,\ 2 \times 6,\ 3 \times 4,\ 4 \times 3,\ 6 \times 2,\ 12 \times 1$ and $(a^2 - b^2) = (a+b)(a-b)$. Take $12 = 12 \times 1,\ 6 \times 2,\ 4 \times 3$. Avoid fractional answers.

(3).

$$4x - 5y = 9$$

$$\Rightarrow \quad 4x = 5y + 9 = 4y + (y+9)$$

$$x = y + \frac{y+9}{4} \quad \text{and} \quad \frac{y+9}{4}$$

should be an integer; *i.e.*, $(y+9)$ should be a multiple of 4. Using tabular column as given below, you can get all integer solutions

y	3	7	11	-1	-5	-9
x	3+3 $= 6$	7+4 $=11$	11+5 $= 16$	$-1+2$ $= 1$	$-5+1$ $= 4$	$-9+0$ $= -9$

You can extend the table: The solutions for (x, y) is (6, 3), (11, 7), (16, 11), (21, 15), $\ldots$ $(1, -1)$, $(-4, -5)$, $(-9, -9)$, $(-14, -13)$.

Note: The values of $x = 6, 11, 16, \ldots$ are increasing by 5, (or decreasing by 5 for $(1, -4, -9, -14)$) and the corresponding values of y is $3, 7, 11, 15, \ldots$ are increasing by 4 (or decreasing by 4, for $-1, -5, -9, -13, \ldots$). The sequence of values of x and y (arranged in ascending order/descending order) is called an AP.

(4).

$$7 + a + b = 10 \Rightarrow a + b = 3$$

The pairs (a, b) satisfying the above equation are (1, 2), (2, 1) (The values for a and b are natural numbers).

(5). Since subtraction is done without carry over, b can take values 0, 1, 2, 3, 4 or 5. The corresponding values of d are 5, 4, 3, 2, 1 and 0 and a can take only 3 values 1, 2, or 3 (if c is non zero). The corresponding values of c are 3, 2 or 1.

$$\begin{array}{cc} 4 & 5 \\ a & b \\ \hline c & d \\ \hline \end{array}$$

The subtraction problems are

(1) $\begin{array}{cc} 4 & 5 \\ 1 & 0 \\ \hline 3 & 5 \\ \hline \end{array}$, (2) $\begin{array}{cc} 4 & 5 \\ 1 & 1 \\ \hline 3 & 4 \\ \hline \end{array}$, (3) $\begin{array}{cc} 4 & 5 \\ 1 & 2 \\ \hline 3 & 3 \\ \hline \end{array}$,

(4) $\begin{array}{cc} 4 & 5 \\ 1 & 3 \\ \hline 3 & 2 \\ \hline \end{array}$, (5) $\begin{array}{cc} 4 & 5 \\ 1 & 4 \\ \hline 3 & 1 \\ \hline \end{array}$, (6) $\begin{array}{cc} 4 & 5 \\ 1 & 5 \\ \hline 3 & 0 \\ \hline \end{array}$

(in Problem 3 c and d take the same value) and in Problem 2, (a, b) take the same value.

(7) $\begin{array}{cc} 4 & 5 \\ -2 & 0 \\ \hline 2 & 5 \\ \hline \end{array}$, (8) $\begin{array}{cc} 4 & 5 \\ -2 & 1 \\ \hline 2 & 4 \\ \hline \end{array}$, (9) $\begin{array}{cc} 4 & 5 \\ -2 & 2 \\ \hline 2 & 3 \\ \hline \end{array}$

(10) $\begin{array}{cc} 4 & 5 \\ -2 & 3 \\ \hline 2 & 2 \\ \hline \end{array}$, (11) $\begin{array}{cc} 4 & 5 \\ -2 & 4 \\ \hline 2 & 1 \\ \hline \end{array}$, (12) $\begin{array}{cc} 4 & 5 \\ -2 & 5 \\ \hline 2 & 0 \\ \hline \end{array}$

If $c = 0$, then repeat all the first 5 problems above, with $a = 4$.

Here, in Problem 9, a and b take the same value and Problem 10, (c, d) take the same value. Continue this and find the total number of subtraction problems that could be constructed.

(6). Do it yourself

(7). Multiplying 28 by a, it gives $20a + 8a$. Thus $2a$ will

$$\begin{array}{ccccc}
 & T & u & & T \quad u \\
 & 2 & 8 & \times & 3 \quad a \\
 & H & T & u & \\
\hline
 & 8 & 4 & & \\
+ & & 2a & + & 8a \\
\hline
\\
\hline
\end{array}$$

be placed in Tens place. Addition has no carry over. Thus

$$2a + 4 + \text{ Carry over from } 8a \leq 9.$$
$$2a + \text{ Carry over from } 8a \leq 5.$$

If $a = 3$, we get $(8 \times 3 = 24)$ and carry over from $8a$ is 3 and $2a = 2 \times 3 = 6 > 5$. Thus, $a \neq 3$, $a = 2$. $2a$ carry over from $8a$ is $4 + 1 = 5$. If $a = 1$, then $2a$ carry over from $8a = 2 + 0 = 2 < 5$. If $a = 0$, then $2a$ carry over from $8a = 0 + 0 = 0 < 5$. Thus a takes 2, 1 or 0 as value. Thus the multiplications 28×30, 28×31, 28×32 have no carry over in the addition part of the problem.

(8) 9 divides $ab5$, $a \neq 0$, a, b and 5 are the 3 digits of the 3 digit number,

$$\therefore a + b + 5 = 9 \text{ or } 18$$
$$a + b = 4 \text{ or } 13$$

$$a+b=4,$$

$$\text{gives} \quad a=1,\ b=3,\ a=2, b=2,\ a=3,\ b=1$$

$$a+b=13,$$

$$\text{gives} \quad a=4,\ b=9,\ a=5, b=8, \cdots a=9,\ b=4.$$

So, the number of 3 digits numbers are $a+b=4$ there are 4 numbers, $a+b=13$, there are 6 numbers. They are 135, 225, 315, 405, 495, 585, 675, 765, 855, 945.
The quotient in each case is (on dividing by 9)
15, 25, 35, 45, 55, 65, 75, 85, 95, 105 (can $a+b+5=27$? Why?)

(9). Adding the 4 numbers we get

$$3(a+b+c+d)=219$$

$$\text{and} \quad (a+b+c+d)=73$$

$$d=73-50=23.$$

$$a+b=58-23=35$$

Solve completely to find the other values.

(10). $\frac{(n+2)(n+2)}{(n-3)}$ is a natural numbers, n is a natural number. Let, $n+2$ be k, so that $(n-3)=k-5$,

$$\frac{(n+2)(n+2)}{(n-3)} = \frac{k^2}{(k-5)}$$

$$k^2 = k^2 - 25 + 25$$

$$= (k-5)(k+5)+25$$

$$\therefore \frac{k^2}{k-5} = (k+5) + \frac{25}{(k-5)}$$

To get $\frac{k^2}{k-5}$ to be a natural number, $(k-5)$ should divide 25. The divisor of 25 are 1, 5 and 25. Thus, $k-5=1$ or 5 or 25 and k

should be 6, 10 or 30 and the corresponding values of n are 4, 8 and 28. Exactly for these 3 values of n, we can have $\frac{(n+2)(n+2)}{(n-3)}$ to be a natural number.

Another method:
By actual division

$$(n+2)^2 \div n-3 = n^2+4n+4 \div n-3$$

$$\begin{array}{r|l}
 & \quad\ \ n+7 \\
\cline{2-2}
n-3 & n^2+4n+4 \\
 & -n^2 \pm 3n \\
\cline{2-2}
 & \quad\ \ 7n+4 \\
 & \quad -7n \pm 21 \\
\cline{2-2}
 & \qquad\ 25 \quad \text{Remainder}
\end{array}$$

or $(n+2)^2 = (n+7)(n+3) + \frac{25}{(n-3)}$ and $\frac{25}{(n-3)}$ should be a natural number. That is $n-3$, should be a divisor of 25 *i.e.*, $n-3$ should be 1, 5, or 25 and hence n should be 4, 8 or 28.

$$(n+2)^2 = 36, \text{ or } 100 \text{ or } 900$$
$$\text{and } \ (n-3) \text{ is } = 1 \text{ or } 5 \text{ or } 25.$$

(11). $x^2+(x+1)^2+(x+2)^2+(x+3)^2+(x+4)^2 = 5x^2+20x+30$
Thus this expression is divisible by 5. The quotient expression is x^2+4x+6 (Remainder is zero).

(12). Do it yourself

(13). Do it yourself

(14).

$$a \star b = 2a+2b-ab$$
$$(4 \star x) \star 5 = (8+2x-4x) \star 5$$

$$
\begin{aligned}
&= (8-2x)\star 5\\
&= (16-4x)+10-(40-10x)\\
&= 16-4x+10-40+10x\\
&= 6x-14\\
3\star(5\star x) &= 3\star(10+2x-5x)\\
&= 3\star(10-3x)\\
&= 6+(20-6x)-(30-9x)\\
&= -4+3x\\
6x-14 &= 3x-4\\
3x &= 10\\
x &= 3\frac{1}{3}
\end{aligned}
$$

(15). Do it yourself

(16). $(a-b)^2+(b-2c)^2+(c-3d)^2+(d-4e)^2=0$, since no square number can be negative, the sum to be zero, we should have all the square numbers to be zero. Thus we have

$$a=b=2c, \quad c=3d, \quad d=4e.$$

$$a+b+c+d+e = 24e+24e+12e+4e+e=130$$

$$i.e., \quad 65e = 130 \quad \text{or}$$

$$e=2, \quad d=8, \quad c=24, b=48, \quad a=48$$

(17).

$$\text{R.H.S} = \frac{1}{n}-\frac{1}{n+1}$$

$$\frac{1}{n+1} = \frac{(n)}{n(n+1)}$$

$$\therefore \frac{1}{n}-\frac{1}{n+1} = \frac{(n+1)-n}{n(n+1)}$$

$$= \frac{1}{n(n+1)}$$
$$= \text{L.H.S}$$

$$\frac{1}{2} = \frac{1}{1\times 2} = \frac{1}{1} - \frac{1}{2}$$
$$\frac{1}{6} = \frac{1}{2\times 3} = \frac{1}{2} - \frac{1}{3}$$
$$\frac{1}{12} = \frac{1}{3\times 4} = \frac{1}{3} - \frac{1}{4}$$
$$\frac{1}{20} = \frac{1}{4\times 5} = \frac{1}{4} - \frac{1}{5}$$
$$\frac{1}{30} = \frac{1}{5\times 6} = \frac{1}{5} - \frac{1}{6}$$
$$\frac{1}{42} = \frac{1}{6\times 7} = \frac{1}{6} - \frac{1}{7}$$

Adding both sides

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} = \frac{1}{1} - \frac{1}{7} = \frac{6}{7}$$

(18).

$$\frac{1}{3} + \frac{a}{b} = \frac{1+a}{4+b}$$
$$\Rightarrow \quad (3a+b)(4+b) = 3b(1+a)$$
$$= 12a + 4b + 3ab + b^2 = 3b + 3ab$$
$$\Rightarrow \quad b + b^2 = -12a$$
$$b(b+1) = -12 \times a.$$

If $b = 0$, $a = 0$; since $\frac{a}{b}$ will be of the form $\frac{0}{0}$, this value is to be rejected.

However if a is 0, then $b \neq 0$, implies $b + 1 = 0$, then $b = -1$. Thus $a = 0, b = -1$ is one value.

Now using the equation $b(b+1) = -12a$, we shall tabulate the integer value for a and b. b and $b+1$ are 2 consecutive numbers and the product is $-$ve. So, excepting for $b = -1$, both b and $b+1$ are positive or both b and $b+1$ are negative. (We already found that $b \neq 0$, therefore (0,1) can not be taken for b and $(b+1)$)

Excepting for $b = -1$, for all the other values of b, a should be negative. Here is the tabular column.

b	-1	3	-4	8	-9	12	-13
$b+1$	0	4	-3	9	-8	13	-12
a	0	-1	-1	-6	-6	-13	13

b	15	-16	20	23	-24	27	-28
$b+1$	16	-15	21	24	-23	28	-27
a	-20	-20	-35	-46	-46	-63	-63

and so on. Infinitely many integer values can be found for (a, b).

(19). $$\left(\frac{x^6+x^6+x^6+x^6}{y^6+y^6+y^6}\right) \times \left(\frac{z^2+z^2+z^2+z^2+z^2+z^2}{k^6+k^6}\right) = 2^n$$

given $x = 2k$ and $z = 2y$. The expression on the left is

$$\frac{4x^6}{3y^6} \times \frac{6z^6}{2k^6} = 4\left(\frac{xz}{yk}\right)^6$$

$$xz = 4ky$$

$$\therefore\ 4\left(\frac{xz}{yk}\right)^6 = 4 \times \left(\frac{4ky}{yk}\right)^6$$

$$= 4 \times 4^6$$

$$= 2^2 \times 2^{12} = 2^{24} = 2^n$$

Therefore $n = 24$

(20). Any two digit prime number is an odd number. So let $(2m+1)$ be the divisor and the dividend be $(2n+1)$, where both $2m+1$ and $2n+1$ are prime numbers. $(2n+1) = (2m+1)q + r$. q and r are the quotient and remainder. If quotient q is even, then $2n+1 =$ an even number$+r$. $\therefore r$ is an odd number and $q+r$ is odd (as sum of even number and odd number). If q is odd, then $2n+1 =$ an odd number $+r$. $\therefore r$ should be even. Again $q+r$ is odd.

(21).

$$(a+b)_{9c} = 9-(a+b)$$
$$a_{9c} = (9-a), \quad b_{9c} = (9-b)$$
$$\text{If} \quad (a+b)_{9c} = a_{9c} + b_{9c}$$
$$\text{then} \quad 9-(a+b) = (9-a)+(9-b)$$
$$\Rightarrow 9-(a+b) = 18-(a+b)$$

which is impossible. There exist no numbers a, b such that $(a+b)_{9c} = a_{9c} + b_{9c}$.

$$a - a_{9c} = 7$$
$$\Rightarrow a-(9-a) = 7$$
$$\Rightarrow 2a - 9 = 7$$
$$\Rightarrow 2a = 16$$
$$\Rightarrow a = 8$$
$$\text{Thus } 8 - 8_{9c} = 7.$$

Note: $a - a_{9c}$ should be an odd number.

(22). Let the digits of the numbers be a and b

Thus $ab + a + b =$ The number is $10a + b$

$$ab + a + b = 10a + b$$

$$ab = 9a, \quad a, \text{ being the 10s digit is} \neq 0$$
$$\therefore\ b = 9.$$

a can be any of 1 to 9 number. Thus 19, 29, 39,...99 are the numbers having this property.

(23). a) $x^2 + 73$, x is an integer, the minimum value is obtained, when $x = 0$,
i.e., $0 + 73 = 73$, is the minimum value. If x is non-zero integer x^2 is positive and $x^2 + 73 > 73$.

b) $x^2 - 6x + 12 = x^2 - 6x + 9 + 3$ and
$x^2 - 6x + 9 = (x-3)^2 + 3$.
But the minimum value of $(x-3)^2 + 3$ is obtained when $(x-3)^2$ is minimum. $(x-3)^2$ is never negative. So when $(x-3)^2 = 0$, (when $x = 3$), the minimum value is obtained and the minimum value is 3.

(24). Try it yourself. *Hint:* $33 + 10x - x^2 = -(x^2 - 10x + 25) + 58$.

(25).

$$\text{Let } x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$
$$= 1 + \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{4} + \dots\right]$$
$$\therefore x = 1 + \frac{1}{2}x$$
$$\Rightarrow \frac{1}{2}x = 1$$
$$x = 2.$$

Use this method to solve the other subdivisions.

(26). Given $x^2 + x + 1 = 0$

$$\text{(i)}\quad x^2 + 1 = -x \tag{14.1}$$

Dividing both sides by x

$$x + \frac{1}{x} = \frac{-x}{x} = -1$$

$$\text{(ii)} \quad x^2 + x = -1$$

$$= x + \frac{1}{x}$$

$$\therefore x^2 = \frac{1}{x} \tag{14.2}$$

$$\text{The reciprocal of } x^2 = \text{The reciprocal of } \frac{1}{x}$$

$$i.e., \ \frac{1}{x^2} = x \tag{14.3}$$

Adding (14.2) and (14.3)

$$x^2 + \frac{1}{x^2} = \frac{1}{x} + x = x + \frac{1}{x} = -1 \quad \text{from eqn (14.1)}$$

$$\text{(iii)} \qquad x^2 = \frac{1}{x} \qquad \text{from eqn (14.2)}$$

$$\Rightarrow x^4 = \frac{x^2}{x} = x \tag{14.4}$$

$$\frac{1}{x^2} = x \qquad \text{from eqn (14.3)}$$

$$\frac{1}{x^2} \times \frac{1}{x^2} = \frac{1}{x^2} \times x$$

$$\Rightarrow \qquad \frac{1}{x^4} = \frac{1}{x} \tag{14.5}$$

$$\therefore x^4 + \frac{1}{x^4} = x + \frac{1}{x} = -1$$

$$\text{(iv)} \qquad x^2 = \frac{1}{x}$$

$$(x^2)^4 = (\frac{1}{x})^4 = \frac{1}{x^4} \tag{14.6}$$

$$\frac{1}{x^2} = x$$
$$\frac{1}{x^8} = (\frac{1}{x^2})^4 = x^4 \tag{14.7}$$

Adding (14.6) and (14.7)

$$x^2 + \frac{1}{x^2} = \frac{1}{x^4} + x^4 = x^8 + \frac{1}{x^8} = -1 \qquad \text{from (iii)}$$

[*Note:* $\frac{1}{x^2} = x \Rightarrow 1 = x^3$ and $x^3 + \frac{1}{x^3} = 1 + 1 = 2$.]

(27). Given the pattern

$$\begin{gathered}
1 = 1^2 \\
1+2+1 = 4 = 2^2 \\
1+2+3+2+1 = 9 = 3^2 \\
1+2+3+4+3+2+1 = 16 = 4^2 \\
1+2+3+4\cdots+(n-2)+(n-1)+ \\
n+(n-1)+(n-2)+(n-3)\cdots+1 = n^2 \\
(1+2+3+\cdots+n)+(1+2+3+\cdots(n-1)) = n^2
\end{gathered}$$

$$\Rightarrow (1+2+3+\cdots+n)+$$
$$(1+2+3+\cdots+(n-1)+n) = n^2+n$$
$$\Rightarrow 2(1+2+3+\cdots+n) = n(n+1)$$
$$\Rightarrow (1+2+3+\cdots+n) = \frac{n(n+1)}{2}$$

(28).

$$(8^{73}+2)^2 - (8^{73}-2)^2 = 8^n$$
$$\Rightarrow [(8^{73})^2 + 2\cdot 8^{73}\cdot 2 + 2^2]$$
$$- [(8^{73})^2 - 2\cdot 8^{73}\cdot 2 + 2^2] = 8^n$$

$$\Rightarrow 2^3 + 8^{73} = 8^n$$
$$\Rightarrow 8^{74} = 8^n$$
$$n = 74$$

Another Method:

$$(a+b)^2 - (a-b)^2 = 4ab$$

Here $\quad a = 8^{73}, \quad b = 2$

$$\therefore 4ab = 4 \cdot 8^{73} \cdot 2$$
$$= 8 \times 8^{73} = 8^{74} = 8^n$$
$$\therefore n = 74.$$

(29). Given $a \times a + b \times b = c \times c$ and $p \times c = a \times b$

$$i.e., \; a^2 + b^2 = c^2 \qquad (14.8)$$
$$\text{and} \quad pc = ab \qquad (14.9)$$

From Eqn. (14.9) $\quad p = \dfrac{ab}{c}$

$$p \times p = p^2 = \frac{a^2b^2}{c^2}$$
$$\therefore \frac{1}{p \times p} = \frac{1}{p^2} = \frac{c^2}{a^2b^2}$$
$$= \frac{a^2 + b^2}{a^2b^2} \qquad \text{from Eqn. (14.8)}$$
$$= \frac{a^2}{a^2b^2} + \frac{b^2}{a^2b^2}$$
$$= \frac{1}{b^2} + \frac{1}{a^2}$$
$$= \frac{1}{b \times b} + \frac{1}{a \times a}$$

$$\text{or} \quad = \frac{1}{a \times a} + \frac{1}{b \times b}$$

$$\text{Thus} \quad \frac{1}{p \times p} = \frac{1}{a \times a} + \frac{1}{b \times b}$$

(30). Given

$$1 = 1 \times 1 \times 1 = 1^3$$
$$3 + 5 = 2 \times 2 \times 2 = 2^3$$
$$7 + 9 + 11 = 3 \times 3 \times 3 = 3^3$$
$$13 + 15 + 17 + 19 = 4 \times 4 \times 4 = 4^3$$

Observing the pattern on the left side, the first row has number 1, which is the first odd number $(2 \times 1 - 1)$ the second row starts with *second odd number* and has *two odd numbers* and ends with 3^{rd} *odd number.* The third row starts with *4th odd number*, and has *3 odd numbers* and ends with *6th odd number.* The fourth row starts with 7^{th} odd number and has *four* odd numbers ends with *10th odd number.*

Thus the last number in each of the rows are 1^{st} odd number, 3^{rd} odd number, 6^{th} odd number, 10^{th} odd number $\cdots$ and hence n^{th} row ends with $\frac{n(n+1)}{2}^{\text{th}}$ odd number (1, 3, 6, 10 , ...are of the form $\frac{n(n+1)}{2}$ for $n = 1, 2, 3, \ldots$) and since n^{th} row has 'n' odd numbers, the first number is $\left(\frac{n(n+1)}{2} - (n-1)\right)^{\text{th}}$ odd number. *i.e.*, $\left(\frac{n^2+n-2n+2}{2}\right)^{\text{th}}$ odd number, or $\left(\frac{n^2-n+2}{2}\right)^{\text{th}}$ odd number which is $\frac{\cancel{2}(n^2-n+2)}{\cancel{2}} - 1 = (n^2 - n + 1)$

(*Note:* $n^2 - n + 1 = n(n-1) + 1$, and $n(n-1)$ is even as, n and $(n-1)$ are consecutive natural numbers and hence $n(n-1) + 1$ is an odd number)

$\therefore$ The number in n^{th} row is

$$(n^2 - n + 1) + (n^2 - n + 3) + (n^2 - n + 5)$$

$$+\cdots+2\times\frac{n(n+1)}{2}-1$$

i.e., $(n^2-n+1)+(n^2-n+3)+(n^2-n+5)+\cdots$
$$\cdots(n^2+n-1)=n^3$$

$$\begin{aligned}
&= n[n^2-n+1]+2+4+6+\cdots 2(n-1)\\
&= n^3-n^2+n+2(1+2+3+\cdots+n-1)\\
&= n^3-n^2+n+\frac{2\times(n-1)n}{2}\\
&= (n^3-n^2+n)+(n^2-n)\\
&= n^3
\end{aligned}$$

which was to be proved.

(31). The given pattern of numbers is ($\sim$ is the symbol for difference.)

1, 2	3	0	$((1+2)\sim 3)$
1, 2, 3	4, 5	3	$(1+2+3)\sim(4+5)$
1, 2, 3, 4	5, 6, 7	8	$(1+2+3+4)$ $\sim(5+6+7)$
1, 2, 3, 4, 5	6, 7, 8, 9	15	$(1+2+3+4+5)$ $\sim(6+7+8+9)$

By observing the pattern, we find that the first column has numbers from 1 to $(n+1)$, in the n^{th} row. The second column of n^{th} row, starts from $(n+2)$ and has n numbers.

i.e., $(n+2), (n+3), (n+4), \ldots \big(n+(n+1)\big)$ $[=(2n+1)]$

and the third column numbers are difference between the sums of the natural numbers in column 1 and 2. For the n^{th} row, the above explanation gives

$(1+2+3\cdots+n+1)\sim$

$$\Big((n+2)+(n+3)+\cdots+(n+(n+1))\Big)$$

$$\Rightarrow \frac{(n+1)(n+2)}{2} \sim$$

$$[(1+2+\cdots+(2n+1))-(1+2+3+\cdots+(n+1))]$$

$$\Rightarrow \frac{(n+1)(n+2)}{2} \sim \left[\frac{(2n+1)(2n+2)}{2} - \frac{(n+1)(n+2)}{2}\right]$$

$$\Rightarrow \frac{(n+1)(n+2)}{2} \sim \frac{3n(n+1)}{2}$$

$$= \frac{3n(n+1)}{2} - \frac{(n+1)(n+2)}{2}$$

$$= \frac{(n+1)}{2}(3n-n-2)$$

$$= \frac{(n+1)}{2}(2n-2)$$

$$= (n+1)(n-1)$$

$$= (n^2-1)$$

(Verify this result writing $n = 1, 2, 3, 4$ and observing the pattern given).

(32). (a) (1) $(a,b) \star (c,d) = (ac+bd, ad+bc)$

$$(5,4) \star (4,5) = (5\times 4 + 4 \times 5, 5\times 5 + 4\times 4)$$
$$= (40,41) = (0,1)$$

(2) $(3,4) \star (4,5) = (3\times 4 + 4\times 5, 3\times 5 + 4 \times 4)$

$$= (32,31) = (2,1) = (1,0)$$

(3) $(6,5) \star (6,5)$ (Do it yourself)

(b)

$$(2,3) \star (a,b) = (2,3)$$

i.e., $(2a+3b, 2b+3a) = (2,3)$

But by definition of equality,
$(2a+3b+3) = (2b+3a+2) \Rightarrow a = b+1$
When $b = 3$, $a = 4$:

$$\text{Hence} \quad (2,3)\star(4,3)(2,3)$$
$$(2,3)\star(4,3) = (2,3)$$

for all natural numbers a.
(c) Given $\quad (a,b)\odot(c,d) = (a+c, b+d)$

$$(2,3)\odot(x,y) = (2+x, 3+y) = (2,3)$$
$$\Rightarrow 2+x+3 = 3+y+2$$
$$\Rightarrow x = y.$$

Thus $(2,3)\odot(a,a) = (2,3)$ for all a where a is a whole number.

$$\begin{aligned}(3,2)\star[(2,3)\odot(3,4)] &= (3,2)\star(5,7)\\ &= (15+14, 21+10) = (29,31)\\ &= (0,2) = \text{L.H.S.}\end{aligned}$$

$$\begin{aligned}\text{R.H.S.} = (3,2)\star(2,3) &= (6+6, 9+4) = (12,13) = (0,1)\\ (3,2)\star(3,4) &= (9+8, 12+6)\\ &= (17,18) = (0,1)\\ (0,1)\odot(0,1) &= (0+0, 1+1)\\ &= (0,2) = \text{L.H.S.}\end{aligned}$$

(33). (a) Let $(2,3) = (x,y)$ then $2y = 3x$ or $y = \frac{3x}{2}$, $y \neq 0$. $\therefore x$ should be an even natural number.
Thus $(2,3) = (4,6) = (6,9) = (8,12) = \cdots = (2a,3a)$, where a is a natural number. Thus there are infinitely many ordered pairs which are equal to one another.

(b)

(i) $\quad (1,2)\star(3,4) = (1\times 4 + 2\times 3, 2\times 4)$

$$= (10, 8) = (5, 4)$$

(viii) $$(5, 4) \star (0, 100) = (500 + 0, 400)$$
$$= (500, 400) = (5, 4)$$

Do the other subdivisions

(c)

(i) $$(4, 3) \odot [(5, 16) \star (7, 16)]$$
$$= (4, 3) \odot [80 + 112, 256] = (4, 3) \odot (192, 256)$$
$$= (4, 3) \odot (3, 4)$$
$$= (12, 12) = (1, 1)$$

(ii) $$[(4, 3) \odot (5, 16)] \star [(4, 3) \odot (7, 16)]$$
$$(20, 48) \star (28, 48)$$
$$\Rightarrow (5, 12) \star (7, 12)$$
$$\Rightarrow (60 + 84, 144) = (144, 144) = (1, 1)$$

This answer is the same as answer for question (i)

(d) (i) $$(3, 4) \odot (a, b) = (3a, 4b) = (3, 4)$$
$$\Rightarrow a = b = 1 \text{ or } (a, b) = (1, 1)$$

(ii)

$$(3, 4) \star (a, b) = (3b + 4a, 4b)$$
$$(3b + 4a, 4b) = (3, 4)$$
$$\Rightarrow \quad 4(3b + 4a) = 12b$$
$$\Rightarrow \quad a = 0.$$
$$\therefore (3, 4) \star (0, b) \quad (\text{for all b}) = (3, 4)$$
$$(3b, 4b) = (3, 4)$$

(iii)

$$(3,4) \odot (a,b) = (3a,4b)$$

If $(3a,4b) = (1,1)$ then $4b = 3a$

$$\Rightarrow \quad (a,b) = (4,3),\ (8,6),\ (12.9),\ \ldots$$

(iv)

$$(3,4) * (a,b) = (3b+4a, 4b)$$

$$\text{or} \quad (3b+4a, 4b) = (0,1)$$

$$3b + 4a = 0$$

$$4a = -3b$$

$$a = -3\times\frac{b}{4}$$

$\therefore$ b must be a multiple of 4.
Any one of $(-3,4) = (-6,8) = (-9,12)$, or
$(3,-4) = (6,-8) = (9,12) = \cdots$
can be the value of (a,b)

(34).

$$(a_1,a_2,a_3) \star (b_1,b_2,b_3) = (a_1+b, a_2 \sim b_2, a_3+b_3)$$

and

$$(a_1,a_2,a_3) \odot (b_1,b_2,b_3) = (a_1b_1, a_2+b_2, a_3b_3)$$

(iii)

$$(4,5,6) * (a,b,c) = (4,5,6)$$

$$\Rightarrow (4+a, 5\sim b, 6+c) = (4,5,6)$$

$$\Rightarrow \quad a = b = c = 0$$

(iv) (a)

$$(4,5,6) \odot \Big[(2,3.4) * (3,4,5)\Big]$$

$$= (4,5,6) \odot (5,1,9) = (20,6,54)$$

(iv) (b) $$[(4,5,6) \odot (2,3,4)] * [(4,5,6) \odot (3,4,5)]$$
$$= (8,8,24) * (12,9,30)$$
$$= (20,1,54)$$

(35). The formulae

(A)
$$(1-x) \times 1 = (1-x)$$
$$(1-x)(1+x) = (1-x^2)$$
$$(1-x)(1+x+x^2) = (1-x^3)$$
$$(1-x)(1+x+x^2+x^3) = (1-x^4)$$
$$(1-x)(1+x+x^2+x^3+x^4) = (1-x^5)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$(1-x)(1+x+x^2+x^3+\cdots+x^{n-1}) = (1-x^n)$$

(B)
$$(1+x) \times 1 = (1+x)$$
$$(1+x)(1-x+x^2) = (1+x^3)$$
$$(1+x)(1-x+x^2-x^3+x^4) = (1+x^5)$$
$$(1+x)(1-x+x^2-x^3+x^4-x^5+x^6) = (1+x^7)$$
$$(1+x) \times (1-x-x^2-x^3-x^4-x^5-x^6-x^7+x^8) = (1+x^9)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$(1+x)(1-x+x^2-x^3+x^4-\ldots+x^{2n}) = (1+x^{2n+1})$$

(C)
$$(1-x) \times 1 = (1-x)$$
$$(1-x) \times (1+x) = (1-x^2)$$
$$(1-x^2) \times (1+x^2) = (1-x^4)$$
$$(1-x^3) \times (1+x^3) = (1+x^6)$$
$$(1-x^4)(1+x^4) = (1-x^8)$$
$$(1-x^n)(1+x^n) = (1-x^{2n})$$

(a)

$$\begin{aligned}1-x^{10} &= (1-x^5)(1+x^5) \qquad \text{From C}\\ &= (1-x)(1+x+x^2+x^3+x^4)\end{aligned}$$

But from (A)

$$(1-x^{10}) = (1-x)(1+x+x^2+x^3+\cdots+x^9) \tag{2}$$

From (1) and (2),

$$\begin{aligned}(1-x^{10}) &= (1-x)(1+x)(1+x+x^2+x^3+x^4)\\ &\quad\times(1-x+x^2-x^3+x^4)\\ &= (1-x)(1+x+x^2+x^3+\cdots+x^9)\\ \Rightarrow\quad &= (1+x)(1+x+x^2+x^3+x^4)\\ &\quad\times(1-x+x^2-x^3+x^4)\\ &= (1-x)(1+x+x^2+x^3+\cdots+x^9), \quad x\neq 1\\ &\quad\times(1+x)(1-x+x^2-x^3+x^4)\end{aligned}$$

(b) $(1+x+x^2+x^3+x^4)(1-x+x^2-x^4)$

$$\begin{aligned}&= \frac{1-x^5}{1-x}\times\frac{(1+x^5)}{1+x} \quad \text{From (A) \& (B)}\\ &= \frac{1-x^{10}}{1-x^2} \quad \text{From (C)}\end{aligned}$$

Let $x^2 = y$. We have

$$\begin{aligned}\frac{1-x^{10}}{1-x^2} &= \frac{(1-(x^2)^5)}{1-x^2}\\ &= \frac{1-y^5}{1-y}\end{aligned}$$

$$=\frac{(1-y)(1+y+y^2+y^3+y^4)}{(1-y)}$$
$$=1+y+y^2+y^3+y^4, \quad y\neq 1$$
$$=1+x^2+x^4+x^6+x^8$$

$$\therefore (1+x+x^2+x^3+x^4)(1-x+x^2-x^3+x^4)$$
$$=(1+x^2+x^4+x^6+x^8)$$

(c)

$$\frac{(1-x^3)}{(1-x)}\times\frac{(1+x^3)}{(1+x)}=\frac{(1-x^6)}{(1-x^2)}$$

Let $x^2=y$ so that $x^6=y^3$

$$\frac{1-x^6}{1-x^2}=\frac{1-y^3}{1-y}=1+y+y^2=(1+x^2+x^4)$$

(36).

$$1-x^8=(1-x^4)(1+x^4)$$
$$=(1-x^2)(1+x^2)(1+x^4)$$
$$=(1-x)(1+x)(1+x^2)(1+x^4). \qquad (1)$$

Again $(1-x^4)=(1-x)(1+x+x^2+x^3)$.

$$1-x^8=(1-x^4)(1+x^4)$$
$$=(1-x)(1+x+x^2+x^3)(1+x^4). \qquad (2)$$

Since (1) & (2) are equal

$$(1-x)(1+x)(1+x^2)(1+x^4)=(1-x)(1+x+x^2+x^3)(1+x^4)$$

Cancelling $(1-x)$ and $(1+x^4)$ from both sides, (Warning! $(1-x)\neq 0$ whenever $(1-x)$ is unsolved in division) we get

$$(1+x)(1+x^2)=(1+x+x^2+x^3)$$

(37).

$$(1-x^6) = (1-x^3)(1+x^3), \quad \text{From (C)} \qquad (1)$$
$$= (1-x)(1+x+x^2)(1+x^3), \quad \text{From (A)}$$
$$(1-x^6) = (1-x)(1+x+x^2+x^3+x^4+x^5) \qquad (2)$$

Since (1) & (2) are equal,

$$(1+x+x^2)(1+x^3) = (1+x+x^2+x^3+x^4+x^5)$$

(Cancelling $(1-x)$ from both sides as $(1-x) \neq 0$)

$$\underbrace{(1-x)(1+x)}(1+x^2) \times (1+x^4)(1+x^8)(1+x^{16})$$
$$= \underbrace{(1-x^2)(1+x^2)}(1+x^4)(1+x^8)(1+x^{16})$$
$$= \underbrace{(1-x^4)(1+x^4)}(1+x^8)(1+x^{16})$$
$$= \underbrace{(1-x^8)(1+x^8)}(1+x^{16})$$
$$= (1-x^{16})(1+x^{16})$$
$$= (1-x^{32})$$

(39). Let $x^2 = y$, so that $x^{32} = (x^2)^{16} = y^{16}$ and $(1-x^2) = (1-y)$.
Now

$$(1-x^{32}) = (1-y^{16}) = (1-y)(1+y+y^2+\cdots+y^{15}$$
$$\frac{1-x^{32}}{1-x^2} = \frac{1-y^{16}}{1-y}$$
$$= \frac{(1-y)(1+y+y^2+\cdots+y^{15})}{(1-y)}, \quad y \neq 1$$
$$= (1+y+y^2+\cdots+y^{15})$$
$$= (1+x^2+x^4+\cdots+x^{30})$$

(40).

$$\underbrace{a+a+\ldots+a}_{\text{x times}} = a^2b$$

$$ax = a^2b$$

$$\therefore x = ab, \quad (a \neq 0)$$

$$\underbrace{b+b+\ldots+b}_{y\text{ times}} = ab^2$$

$$by = ab^2$$

$$y = ab = x, \quad (b \neq 0)$$

$$\underbrace{(x+x+x\cdots+x)}_{y\text{ times}} + \underbrace{(y+y+\cdots+y)}_{x\text{times}}$$

$$\Rightarrow xy + xy = 2xy$$

$$\Rightarrow 2 \times ab \times ab = 2a^2b^2 \quad (\text{as } x = y = ab)$$

(41).

$$\frac{a}{b} = \frac{c}{d} = k, \quad \text{say}$$

$[a, b, c, d, p, q, r, s$ are natural numbers$]$

$$a = bk, \quad c = dk$$

$$\frac{pa+qb}{ra+sb} = \frac{pbk+qb}{rbk+sb} = \frac{b(pk+q)}{b(rk+s)} = \frac{pk+q}{rk+s}.$$

$$\frac{pc+qd}{rc+sd} = \frac{pdk+qd}{rdk+sd} = \frac{d(pk+q)}{d(rk+s)} = \frac{pk+q}{rk+s}$$

$$\therefore \quad \frac{pa+qb}{ra+sb} = \frac{pc+qd}{rc+sd}.$$

(42). Let the consecutive numbers be a and $a+1$. Their sum is $2a+1$
Thus the three numbers are

a, $a+1$, $2a+1$. If 3 does not divide a or $(a+1)$, then $a = 3k+1$, $a+1 = 3k+2$. The sum

$$\begin{aligned} 2a+1 &= 3k+1+3k+2 \\ &= 6k+3 \\ &= 3(2k+1) \end{aligned}$$

Thus, if neither of the consecutive numbers is divisible by 3, then, their sum is divisible by 3. If the sum $a+a+1 = 2a+1$ is not divisible by 3, then

$$2a+1 = 3k+1 \quad \text{or} \quad 3k+2$$

$$\text{If} \quad 2a+1 = 3k+1, \quad \text{then} \quad 2a = 3k.$$

Here 3 and 2 are relatively prime and so $3 \mid a$.

$$\text{If} \quad a+a+1 = 2a+1 = 3k+2, \quad \text{then}$$

$$2a+2 = 3k+3 = 3(k+1)$$

$$\therefore \quad 2(a+1) = 3(k+1)$$

$$\therefore \quad 3 \mid (a+1).$$

Hence either a, $a+1$ or $2a+1$ is divisible by 3.

(43).

$$\begin{aligned} (a^2+b^2)^2 &= (a^2)^2 + 2 \times (a^2) \times (b^2) + (b^2)^2 \\ &= a^4 + 2a^2 \times b^2 + b^4 & (1) \\ (a^2-b^2) &= a^4 - 2a^2b^2 + b^4 & (2) \\ (2ab)^2 &= 4a^2b^2 & (3) \\ (2)+(3) &= (a^4 - 2a^2b^2 + b^4) + 4a^2b^2 \\ &= a^4 + 2a^2b^2 + b^4 \end{aligned}$$

same as given (1).

Note: If a and b are relatively prime, one odd and the other even,

then a^2+b^2, a^2-b^2 and $2ab$ are also relatively prime. There is no common factor dividing a^2+b^2, a^2-b^2 and $2ab$. Further two of them have a common factor. We get the pairs (7, 2), (9, 6), (23, 22) for (a, b). The PPTs are (53, 45, 28), (117, 45, 108) and (1013, 45, 1012).

(44).

$$\begin{aligned} a^2-b^2 &= (a-b)(a+b) \\ &= 45 \\ &= 5\times 9, \quad 3\times 15, \quad 1\times 45 \end{aligned}$$

(45).

$$\begin{aligned} a^2-b^2 &= (a-b)(a+b) \\ &= 105 \\ &= 1\times 105 = 3\times 35 = 7\times 15 = 5\times 21 \end{aligned}$$

Solve for a and b. You get 4 primitive Pythagorean Triples.

(46). $4984^{2005}+1027^{2005}+978^{2005}-2979^{2005}$
$=(4984^{2005}-2979^{2005})+(1027^{2005}+978^{2005})$
By the algebraic identities given in Question 35,

$$\begin{aligned} x^{2n+1}+y^{2n+1} &= (x+y)(x^{2n}-x^{2n-1}\cdot y+\cdots+y^{2n}) \\ \text{and } x^{2n+1}-y^{2n+1} &= (x-y)(x^{2n}+\cdots+y^{2n}) \end{aligned}$$

Thus for odd powers, $x^{2n+1}+y^{2n+1}$ is divisible by $x+y$, and $x^{2n+1}-y^{2n+1}$ is divisible by $(x-y)$.
So $(4984^{2005}-2979^{2005})$ is divisible by $(4984-2979)=2005$ and also $(1027^{2005}+978^{2005})$ is divisible by
$(1027+978)=2005$. Thus the expression is divisible by 2005.

(47). $2005 = 5 \times 401; 401$ and 5 are relatively prime.

$$= 1562^{2n+1} - 636^{2n+1} + 1646^{2n+1} - 567^{2n+1}$$
$$= (1562^{2n+1} + 1646^{2n+1}) - (636^{2n+1} + 567^{2n+1})$$

and $1562^{2n+1} + 1646^{2n+1}$ is divisible by $1562 + 1646 = 3208 = 401 \times 8$, and $636^{2n+1} + 567^{2n+1}$ is divisible by $636 + 567 = 1203 = 401 \times 3$. Thus the given number is divisible by 401. Similarly $(1562^{2n+1} - 567^{2n+1})$, $(1646^{2n+1} - 636^{2n+1})$ are divisible by 1562-567=995 and 1646-636=1010 respectively. This implies that this number is also divisible by 5. Hence the given number is divisible by $5 \times 401 = 2005$ for all values of n.

(48).

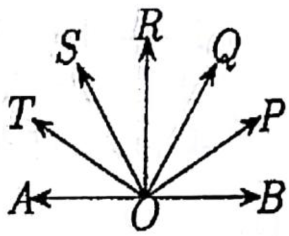
$$ab = cd \text{ and}$$
$$a - b > c - d$$
$$\Rightarrow (a-b)^2 + 4ab > (c-d)^2 + 4cd$$
$$\Rightarrow (a+b)^2 > (c+d)^2$$
$$\Rightarrow a + b > c + d$$

(49). Do it yourself

(50). Do it yourself

Chapter 15

Geometry



(1). Since $\angle BOP = \angle POQ$ $\cdots = \angle TOA$, and since sum of these angles $= 180°$, each angle is $30°$.

a) Count the number of $\angle$s and name them. Count number of $30°$, $60°$, $90°$, $120°$, $150°$, $180°$ angles. It is 21 including the straight $\angle AOB$.

b) Count the acute $\angle$s (*i.e.*, $30°$ and $60°$ angles).

c) The number of right angles angles are 4 ($\angle BOR$, $\angle POS$, $\angle QOT$, $\angle ROA$).

d) Count the obtuse angles.

(2). Do it yourself

(3). Since each of the smaller angle is less than $45°$, their sum is less than $90°$. So the third angle is greater than $90°$, and hence the triangle is obtuse.

(4). Since each of the smaller angles is greater than $45°$, sum of these should be greater than $90°$. So the third angle should be less than $90°$ (Why?). Also

this 3^{rd} angle is the greatest angle. So, the first two should be less than this third angle, which is acute. Hence all the three angles are acute and hence it is an acute angled triangle.

(5). Each of the smaller angles is equal to half the biggest angle.

Let the measure of the biggest angle be x.
Each of the smaller angles is $\frac{1}{2}x$.

$$\text{Sum of these angles} = \frac{1}{2}x + \frac{1}{2}x + x = 180^\circ$$

$$\begin{aligned} i.e., \quad 2x &= 180^\circ \\ \text{Or} \quad x &= 90^\circ \\ \frac{1}{2}x &= 45^\circ. \end{aligned}$$

So the angles of the triangle are 45°, 45° and 90°.

(6). Let the smaller angle be x. Each of the bigger angles is $2x$.

$$\begin{aligned} \text{Sum of the angles} \quad 5x &= 180^\circ \\ x &= 36^\circ \\ 2x &= 72^\circ \end{aligned}$$

Angles of the triangle are 72°, 72° and 36°

(7). The angle measures are *integer valued in degree measure*. The triangle is obtuse angled.
Exactly one of the angles is double the other angle.
First let us assume that, the two angles, one of which is twice the other, are acute angles. So the third angle is obtuse.
Case I:
i.e., x, $2x$ and y are the these angles,

$$y > 90^\circ$$

$$\text{then} \quad 3x < 90°$$
$$x < 30°$$

Since x is an integer, the value of x ranges from 1 to 29. Thus there are 29 triangles.

Have a tabular column to find the measures of the there angles as shown here:

S.No.	x	$2x$	y
1	29°	58°	93°
2	30°	60°	90°
	⋮	⋮	⋮
29	1°	2°	177°

Case II:
If x and y are acute and $2x$ is obtuse, then

$$2x > 90° \Rightarrow x > 45°$$
$$\text{and also} \quad 3x < 180° \Rightarrow x < 60°$$
$$\text{Since,} \quad x + y < 90°$$
$$y < (90 - x) = 90 - 45 = 45°$$
$$\text{and} \quad 45° < x < 60°$$

Again have a tabular column.

x	$2x$	y
46°	92°	42°
⋮	⋮	⋮
59°	118°	3°

Give reason: Can y be 44°?
Complete the table and give the complete solution.

(8). In a right angled triangle, if the right angle is twice the other, then one angle is 90° and each of the other two angles are 45°.
If one acute angle is twice the other acute angle, measuring x°, then sum of the acute angles is $x + 2x = 90^\circ$

$$x = 30^\circ$$
$$2x = 60^\circ$$

So the angles of the triangles are 30°, 60° and 90° and 45°, 45° and 90°.

(9). Let the three angles be x°, $2x^\circ$ y°.
Neither $x = y$, nor $2x = y$.
Since $3x + y = 180^\circ$, we get $y < 90 \Rightarrow 3x > 90^\circ \Rightarrow x > 30^\circ$. Also $2x < 90$, $x < 45$.
That is, the value of x lies between 30° and 45°.
As before, construct the tabular column and solve.

x°	$2x^\circ$	y°
31°	62°	87°
⋮	⋮	⋮
44°	88°	48°

[There are 14 Δs with 2 pairs of Δs having exactly one common angle. Complete the table and verify.]

(10). (b) The perimeter $a + b + c = 13$ cm.
$a + b > c$; $(a - b) < c$; $a + b + c > 2c$
i.e., $13 > 2c \Rightarrow c \leq 6$.

S.No	$(a+b)$	a	b	c	
1	7	1	6	6	✓
2	7	2	5	6	✓
3	7	3	4	6	✓
4	8	1	7	5	✗
5	8	2	6	5	✓ ⋆⋆
6	8	3	5	5	✓
7	8	4	4	5	✓
8	9	1	8	4	✗
9	9	2	6	4	✗
10	9	3	7	4	✗
11	9	4	5	4	✓ ⋆⋆
12	10	1	9	3	✗
13	10	2	8	3	✗
14	10	3	7	3	✗
15	10	4	6	3	✓ ⋆⋆
16	10	5	5	3	✓ ⋆⋆
17	11	6	5	2	✓ ⋆⋆
18	11	7	4	2	✗
19		6	6	1	✓ ⋆⋆

The lengths marked ✗ will not give a Δ. The ⋆⋆ mark denotes repetition. You can now count the number of Δs with perimeter 13 to be 5, of which three are isosceles (6, 6, 1), (4, 4, 5) and (3, 5, 5).
Do the other problems.

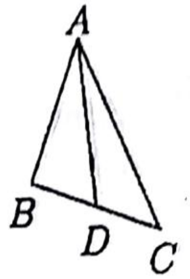
(11). (a) x, y, z are three positive integers.

$$(x+y)+(y+z) = x+2y+z > x+z$$
$$(x+y)+(x+z) = (2x+y+z) > y+z$$
$$(y+z)+(z+x) = (2z+x+y) > x+y$$

Thus, the sum of any two of them is greater than the third. Hence the result.

(b) Do it yourself.

(12). *Given:* ABC is a triangle and D is the mid-point of BC and AD is drawn. (AD is called the median from A to BC.) (In any Δ, sum of the lengths of any two sides is greater than the length of the third side). In $\Delta\ ADB$,



$$AD + DB > AB, \tag{1}$$

$$AD + DC > AC \tag{2}$$

Adding (1) and (2)

$$2AD + DB + DC > AB + AC$$

$$\Rightarrow 2AD + BC > AB + AC$$

$$\Rightarrow AD > \frac{AB + AC - BC}{2}$$

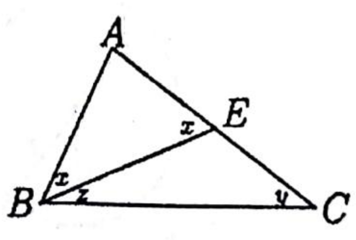
(13). Do it yourself.

(14). Combine the results of Problems 12 and 13 to get the solution for Question 14.

(15). In any triangle ABC if $AB < AC$, then $\angle C < \angle B$ (or $AC > AB \Rightarrow \angle B > \angle C$)

Proof: On AC, mark E, such that $AB = AE$ (It is possible as $AC > AB$).

In ΔABE, $\angle ABE = \angle AEB = (x)$ say, $AB = AE$.



If $\angle EBC = z$, and $\angle ECB = y$, then $x = \angle AEB = (z + y)$. (Give reason.)

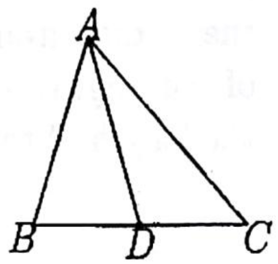
$$\angle ABC = x + z = z + y + z > y = \angle ACB$$

i.e., $x + z > z + y$, $\quad \therefore x + z > y$

or $\quad y < (x + z)$ (Give reason.)

i.e., $\angle ACB < \angle ABC$ or $\angle C < \angle B$.

(16). *Given* AD is the median, and $AD > DC$ ($=DB$).



$$\angle DCA > \angle DAC. \qquad (1)$$

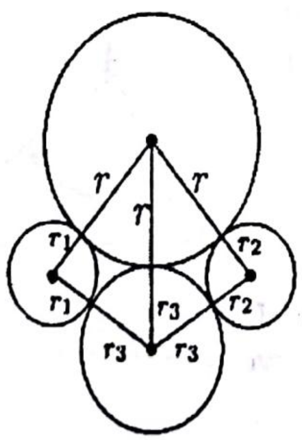
$$AD > DB,$$

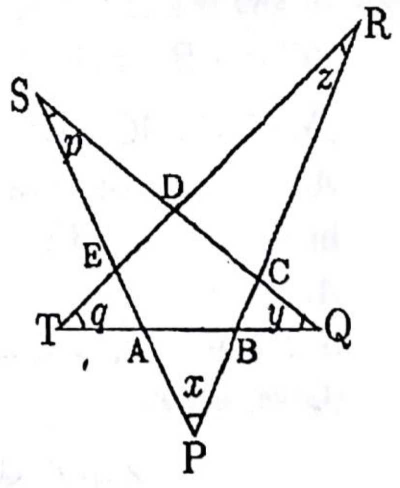
$$\angle DBA > \angle DAB \qquad (2)$$

Adding (1) and (2), $\angle C + \angle B > \angle A$. If $A \geq 90°$, then $\angle B + \angle C > \angle A \geq 90° \Rightarrow \angle B + \angle C > 90°$
$\therefore \angle A + \angle B + \angle C > 90 + 90 = 180$. But the sum of the three angles of a triangle is equal to $180°$. $\therefore \angle A \not\geq 90°$. i.e., $\angle A < 90°$ or $\angle A$ is acute.

(17). Another method (easier!) where the centres A, B and C are not collinear.

Hint: Look at the following figures.





Consider Δs TRB, PSC, BCQ.

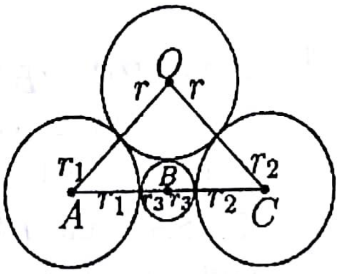
$$\angle RBQ = q^\circ + z^\circ \quad \text{(Why?)} \tag{1}$$

$$\angle PCQ = x^\circ + p^\circ \tag{2}$$

$$\angle RBQ + \angle PCQ + \angle BQC = q^\circ + z^\circ + x^\circ + p^\circ + y^\circ$$

$$= 180^\circ \quad \text{three angles of } \Delta BCQ$$

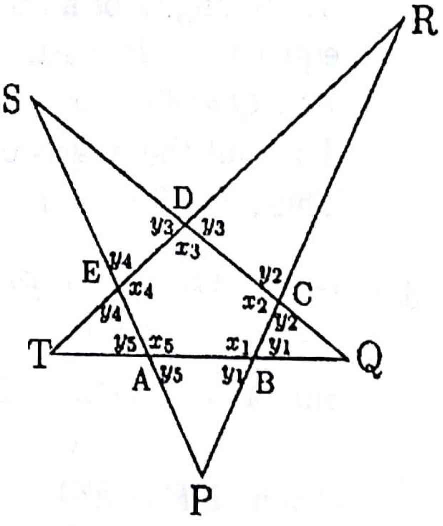
(18). Refer to the figure alongside.



In ΔOAC, $OA + OC > AC$, $\Rightarrow 2r + r_1 + r_2 > r_1 + 2r_3 + r_2$, $\Rightarrow 2r > 2r_3$, $\Rightarrow r > r_3$

(Here the lines of centres AB and BC lie on the same line). Try to prove that r is greater than at least one of the radii for the figure, where A, B, C are not collinear.

(19). $ABCDE$ is a pentagon. All the sides are extended both ways as shown, giving rise to the five vertex star $PQRST$ and five Δs ABP, BCQ, CDR, DES and EAT.



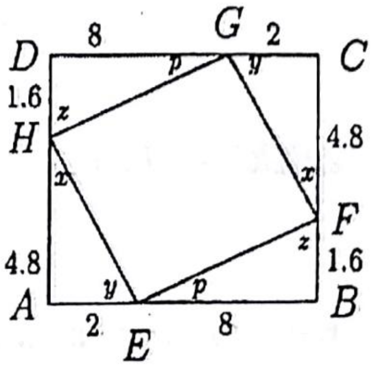
Sum of all the $\angle$s of the pentagon is $3 \times 180^\circ = 540^\circ$. Sum of all the 10 base angles of all the 5Δs got out side the pentagon $= 10 \times 180^\circ - 2(A + B + C + D + E) = 1800 - 2 \times 540^\circ = 1800 - 1080 = 720^\circ$

Sum of all the $\angle$s of the 5 triangles $= 5 \times 180 = 900^\circ$.

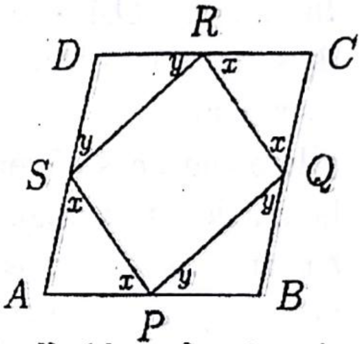
$\therefore$ sum of the $\angle$s at the vertices of the star$= 900 - 720 = 180^\circ$.

Note: For any n sided polygon $n > 4$, we get n such vertices. The sum of the $\angle$s at the vertices $= (n \times 180^\circ) - [2n \times 180 - 2(n-2) \times 180^\circ] = 180n - 720^\circ = 180(n-4)^\circ$

Problems 20 to 32: Do it yourself.



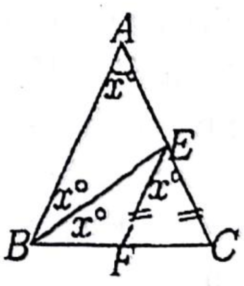
(33). Δs AEH and CGF are congruent.
$\therefore\quad HE = GF.$
Δs BEF and DGH are congruent.
$\therefore\quad EF = HF.$
$\therefore\quad EFGH$ is a parallelogram.



(34). $AP = PB = BQ = QC = CR = RD = DS = SA = \frac{1}{2}AB = \frac{1}{2}BC = \frac{1}{2}CD = \frac{1}{2}AD.$
Δs SAP and RCQ are congruent and Δs SDR and QBP are congruent. ($\angle A = \angle C$, $\angle B = \angle D$, opposite angles of a rhombus are equal; all sides of a rhombus are equal.) $\therefore SP = RQ$; $RS = PQ$.
$\angle SPQ = 180^\circ - x - y = \angle PQR = \angle QRS = \angle RSP.$
Thus all the angles of the $PQRS$ are equal, each equal to 90°.
Thus, $PQRS$ is a rectangle.

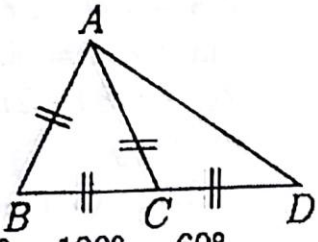
(35). In Δs APS and CRQ, $AP = CR$, $AS = CQ$ and $\angle SAP = \angle QCR$. $\therefore \Delta APS \equiv \Delta CRQ$. Hence $PS = QR$.
Similarly $\Delta BPQ \equiv \Delta DRS$ and hence $SR = PQ$.

(36). Given: $EF = EC$
$\therefore \angle ACB = \angle EFC = \frac{180 - x}{2} = 90 - \frac{x}{2}$
$\angle A + \angle B + \angle C = x + 2x + 90 - \frac{x}{2} = 180^\circ$
$\Rightarrow \frac{5}{2}x = 90^\circ \Rightarrow x = 36^\circ$
$\therefore 2x = 72$, and $\angle C = 90 - \frac{x}{2} = 90 - 18 = 72^\circ$ and hence $AB = AC$ and ΔABC is isosceles.



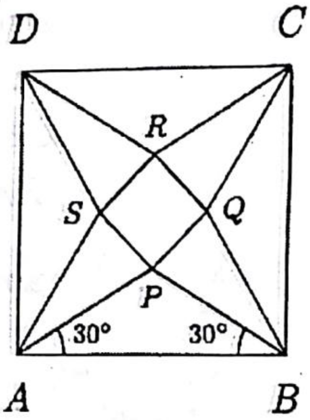
In ΔABC, $AB = AC = BC$.

(37). $\therefore \Delta\ ABC$ is equilateral



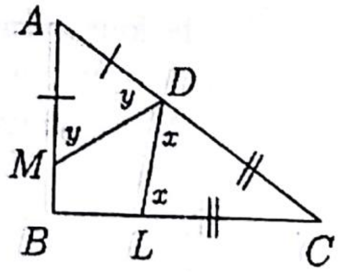
$\therefore \angle BAC = \angle ABC = \angle ACB =$
$60^\circ\ \angle ACD = 180^\circ - 60^\circ = 120^\circ$
In $\Delta\ ACD, AC = CD$.
$\therefore \angle CAD = \angle CDA$.
$\angle CAD + \angle CDA = 180 - \angle ACD = 180^\circ - 120^\circ = 60^\circ$
$\therefore \angle CAD = \angle CDA = \frac{60}{2} = 30^\circ$.
$\therefore \angle BAD = \angle BAC + \angle CAD = 60^\circ + 30^\circ = 90^\circ$
$\angle ABC = 60^\circ. \angle ACD = 120^\circ, \angle BAD = 90^\circ$

(38). $\angle APB = 180^\circ - 60^\circ = 120^\circ$



$= \angle BQC = \angle CRD = \angle DSA$
In Δs APB & ASD, $AB = AD$,
$\angle SAD = \angle PAB = 30^\circ$
$\angle SDA = \angle PBA = 30^\circ$.
Δs are congruent (ASA)
$\therefore AS = DS = AP = PB$
$(= BQ = QC = CR = DR)$.
Also $PQ = QR = RS = SP$. (How?)
$\angle APB = \angle ASP = \frac{180^\circ - 30^\circ}{2} = 75^\circ$
$= \angle BPQ = \angle BQP = \angle CQR = \angle CRQ = \angle DRS = \angle DSR$.
$\therefore \angle SPQ = 360^\circ - (\angle APB + \angle APS + \angle BPQ)$
$= 360^\circ - (120^\circ + 75^\circ + 75^\circ) = 360^\circ - 270^\circ = 90^\circ$
and $\angle PQR = \angle QRS = \angle RSP = 90^\circ$.
In the quadrilateral $PQRS$, $PQ = QR = RS = SP$, and $\angle P = \angle Q = \angle R = \angle S = 90^\circ$ and hence $PQRS$ is a square.

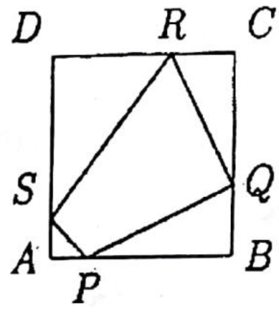
(39). In the figure ABC is a right angled Δ.



i.e., $\angle B = 90^\circ$, $AM = AD$,
$CD = CL$, $AM = AD$
$\Rightarrow \angle AMD = \angle ADM = y$, say.
Similarly $LC = DC$
$\Rightarrow \angle CDL = \angle CLD = X'$, say.

$\angle A = 180 - 2Y$, $\angle 180 - 2x$.
Since $\angle B = 90°$, $\angle A + \angle C = 180° - 90° = 90° = (180 - 2y) + (180 - 2x) \Rightarrow 90 = 360 - 2(x + y)$
$\Rightarrow x + y = \frac{360-90}{2} = 135°$. $\therefore \angle MDL = 180° - x - y$
$= 180 - (x + y) = 180° - 135° = 45°$

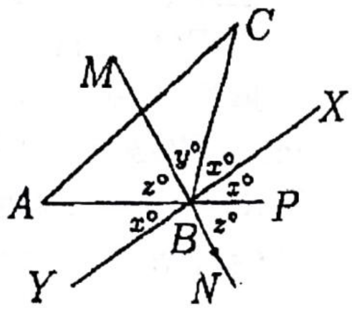
(40). In any Δ, sum of the lengths of two sides is greater than the third side.



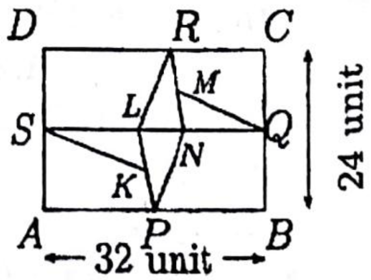
$$\therefore DR + DS > SR$$
$$RC + CQ > RQ$$
$$QB + BP > PQ$$
$$PA + AS > PS$$

Adding the above, we get
$(DP + PC) + (CQ + QB) + (BR + RA) + (AS + DS)$
$> SP + PQ + QR + SR$
$AB+BC+CD+DA > SP+PQ+QR+SR$
i.e., perimeter of quadrilateral $PQRS$ is less than perimeter of square.

(41). $\angle XBM = 90° = \angle YBM$
$(MN \perp XY)$
i.e., $x + z = x + y$
$\therefore z = y$. *i.e.*, $\angle ABM = \angle CBM$.
i.e., BM bisects $\angle ABC$.



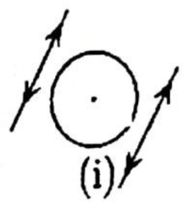
(42). The four triangles PLN, SLK, RNL, QNM are congruent. Let $LN = LK = NM = x$ units. S, L, N, Q are collinear. Then
$SL = QN = \frac{32-x}{2} = 16 - \frac{x}{2} = PL = RN$.



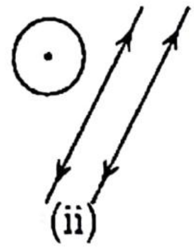
But $PL + RN = AD = 24$.
i.e., $(16 - \frac{x}{2}) + (16 - \frac{x}{2}) = 24 = 32 - x = 24$ *i.e.*, $x = 8$.
Thus $SL = 16 - \frac{8}{2} = 12$.
$\therefore$ Area of each $\Delta = \frac{1}{2} \times SL \times KL = \frac{1}{2} \times 12 \times 8 = 48$ sq. units

(43). The two lines are parallel and

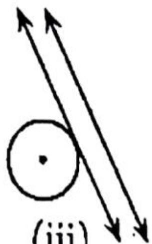
(a) the circle lies between them, without touching them.


(i)

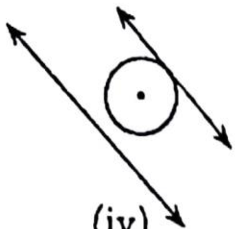
(b) The parallel lines lie on the same of the circle without touching the circle. In (a) and (b) there is no point of intersection.


(ii)

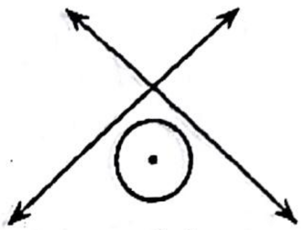
(c) One line touching the circle (tangent to the circle), the other line parallel to it, on the same side of the circle as the touching line, lying further away from the circle. Here also there is one point of intersection.


(iii)

(d) The figure (iv) is yet another configuration giving one point of intersection.

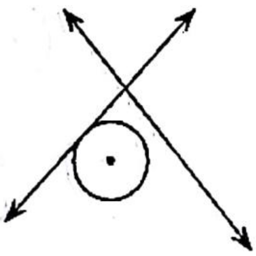

(iv)

(e) The circle lies between the angle made by the intersecting lines determining just one point of intersection.

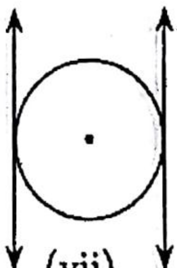


(v)

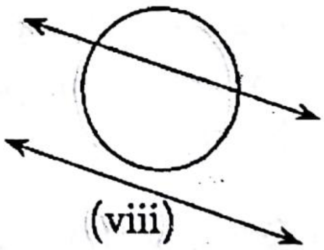
(f) The figures below give configuration giving 2, 3, 4 and 5 points of intersection, 5 is the maximum number of points of intersection that can be obtained when a circle and two lines are given.



(vi)



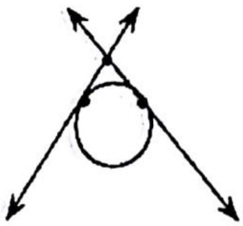
(vii)



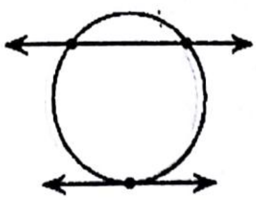
(viii)

Two points of intersection

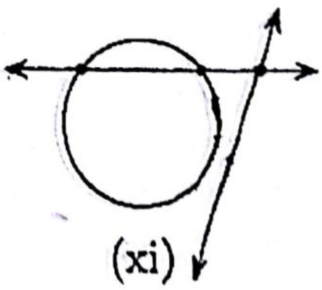
(g)



(ix)



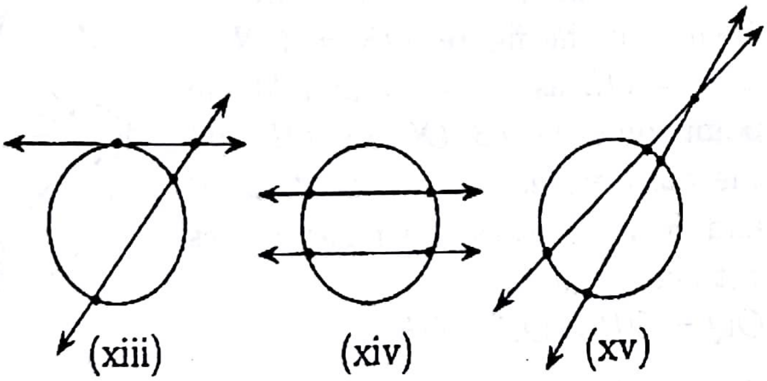
(x)



(xi)

Three points of intersection

(h)



(xiii) (xiv) (xv)

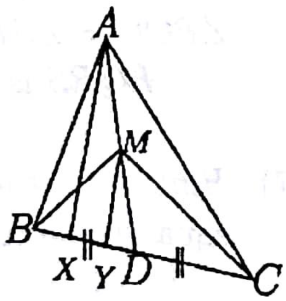
Four points of intersection

You can draw one more figure to get 5 points draw it.

(44). Do it yourself.

(45). If AX is the altitude of $\Delta\ ABC$ Area of $\Delta ABD = \frac{1}{2}BD \times AX = \frac{1}{2}CD \times AX$

$= \text{Area of } \Delta ACD$ (1)



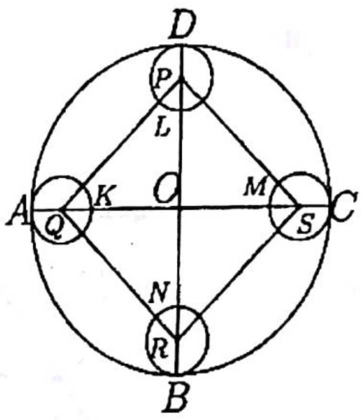
If MY is the altitude of $\Delta\ BMC$,

$$\text{area of}\Delta BMD = \tfrac{1}{2}BD \times MY$$
$$= \tfrac{1}{2}DC \times MY$$
$$= \text{Area of}\Delta MDC \qquad (2)$$

From eqns (1) and (2) we get,

$$\text{Area of}\,\Delta ABD - \text{Area of}\,\Delta BMD$$
$$= \text{Area of}\,\Delta ACD - \text{Area of}\,\Delta CMD \quad (3)$$

Eqn. (3) implies Area of ΔAMB = Area of ΔAMC



(46). To prove that $PQRS$ is a square.
Proof: In the figure, $OK = ON = OM = OL$ as L, K, N and M are points on OA, OB, OC and OD and the radii are equal. Since, P, Q, R and S are centres of smaller circles with equal radii,
$OQ = OR = OS = OP$.

Since $\angle QOR = \angle ROS = \angle SOP = \angle POQ = 90°$,
the Δs QOR, ROS, SOP and POQ are congruent. [by SOS].
$\therefore QR = RS = SP = PQ$
Since OQ, OR, OS and OP are equal,
$\angle OQR = \angle ORQ = \angle ORS = \angle OSR$
$= \angle OSP = \angle OPS = \angle OQP = 45°$ and hence
$\angle RQO = \angle SRO = \angle PSO = \angle QPO = 45 + 45 = 90°$.
$\therefore$ $PQRS$ is a square.

(47). *Hint:* $\angle$s made by PQ, QR ..., UP at the centre O are all equal, equal to 60°.

(a) $\therefore \Delta$s OPQ, OQR, ORS, OST and OTU are equilateral.
$\therefore$ $OP = PQ = QR = RS = ST = TU = UP$.
$\angle PQR = \angle QRS = \angle UPQ = 60° + 60° = 120°$ and hence $PQRSTU$ is a regular hexagon.

(b) $OP = PQ = QR = RO$; hence $OPQR$ is a rhombus.
Note: $\angle POR = 60° + 60° = 120°$ $(\neq 90°)$

(c) $OU = OF - FU = OA = OP$. (OF, OA radii of bigger circle. FU, AP are radii of equal smaller circles.)

(d) $\angle PUT = 120°$ $\angle UPO = 60°$.

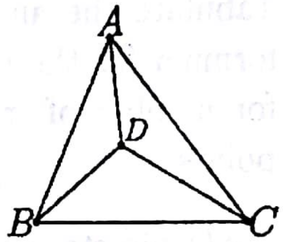
$$\therefore UT \parallel PO,$$

i.e., $UT \parallel PS$ and hence $PUTS$ is a trapezium. [In fact an isosceles trapezium as $PU = TS$. Also, $PS = PO + OS = UT + UT = 2UT$.]

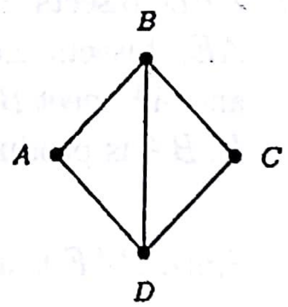
(e) Δs OPR, OPT and OTR are congruent. (Give reason.) $\therefore PR = PT = TR$ and hence $\Delta\ PTR$ is equilateral.

(f) $\angle POR = \angle POQ + \angle QDR = 60° + 60° = 120°$. Also $OP = OR$. $\therefore$ ΔOPR is isosceles. Hence $\angle OPR = \angle ORP = \frac{180-120}{2} = 30°$.

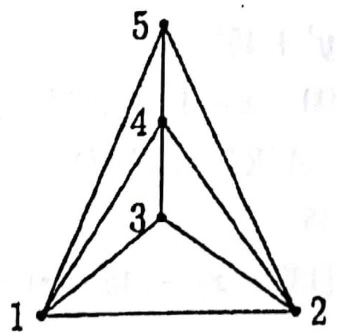
(48). Look at the adjoining figure. Six line segments can be drawn, and this is the maximum number.



(Caution: This figure gives just 5 non intersecting segments; but it is not the maximum.)



(49). The figure that gives maximum line segments for 5 points is the following figure. (*Note:* Two line segments here, have the vertex as the common point. They do not cross each other. The second figure shows the position of the points.)

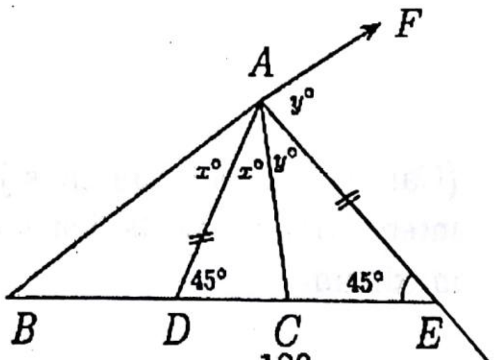


Here are 9 line segments. There are other configurations, they do not give maximum line segments. For six points, you have 12 line segments (draw the figure). Draw figure for 7 points and count the number of line segments; For n points, you get $(n-2)3 = (3n-6)$ line segments.

(When a new point is introduced 3 new line segments are obtained. (Prove !) Use it to prove that there are $3(n-2)$ non intersecting line segments when n points are given)

(50). Draw the figures for $n = 4, 5, 6, 7$ points and count the Δs. Tabulate the answers for $n = 4, 5, 6, 7$. Derive the general formula for the maximum number of Δs. The general formula for number of triangles is $3m - 8$, where m is the number of points.

(51). $\angle AD$ bisects $\angle BAC$ and AE, bisects $\angle CAF$, AD and AE meet BC at D and E, BA is produced to F.



Since BAF is a line

$$2x° + 2y° = 180° \therefore x + y = \frac{180}{2} = 90°$$

$$i.e., \angle DAE = (x° + y°) = 90°$$

$$AD = AE \quad \text{given}$$

$$\therefore \angle ADE = \angle AED = 45°$$

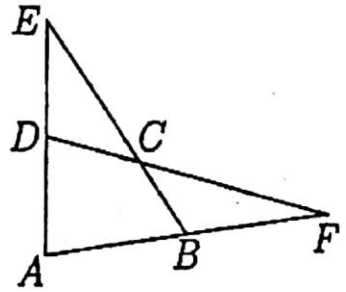
$$\angle ACB = y° + 45°$$

$$= 90 - x + 45 = 135° - x°$$

$$\angle ABD = \angle ADC - \angle BAD$$

$$= 45 - x$$

$$\angle ACD - \angle ABC = (135 - x) - (45 - x) = 90°$$



(52). To prove
$\angle DEC + \angle CFB = \angle DCB - \angle DAB$
Proof:

$$\angle DEC = \angle AEB$$
$$= 180^\circ - (\angle EAB + \angle EBA)$$
$$= 180 - (\angle DAB + \angle CBA) \quad (1)$$

$$\angle BFC = \angle AFD$$
$$= 180 - (\angle FAD + \angle FDA)$$
$$= 180 - (\angle BAD + \angle CDA) \quad (2)$$

Adding (1) & (2)

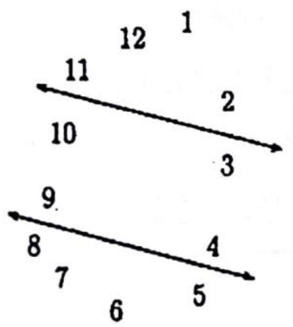
$$\angle DEC + \angle BFC = 180 - (\angle DAB + \angle CBA)$$
$$+ 180 - (\angle BAD + \angle CDA)$$
$$= 360 - (\angle A + \angle B + \angle A + \angle D)$$
of quadrilateral $ABCD$
$$= \underbrace{360 - (\angle A + \angle B + \angle D)} - \angle A$$
$$= \angle C - \angle A$$
(of quadrilateral $ABCD$)

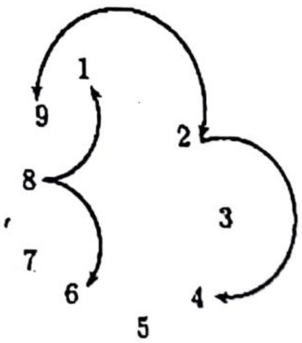
(53). Do it yourself.

Chapter 16

Miscellaneous

(1). Arrange the numbers 1 to 12 in a circle as in a clock, as shown hare. You get the three groups as $(1, 2, 11, 12)$, $(3, 4, 9, 10)$, $(5, 6, 7, 8)$. Here sum of three such groups $= 1 + 2 + \cdots = 12 = \frac{12\times13}{2} = 78$ $\therefore$ Each group must have a sum 26. A second such group is $(11, 10, 2, 3)$, $(12, 1, 9, 4)$, $(5, 6, 7, 8)$.





(2). One such grouping is (9, 2, 4), (1, 8, 6), (3, 5, 7). (Sum of each group is 15.) For 9, 8 and 7, the largest number must go to a different group. For the total to be 15, the group with 9 must have either 1, 5 or 2, 4 (counted earlier). (9, 1, 5) is one group. (8, 3, 4) is the second group and (7, 2, 6) is the third group. It has only two groupings.

(3). For each number there are two choice of colours red or green. Thus, the total number of ways in which the numbers can be written is $2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$. But these 32 different ways include, using red for all numbers and using green for all numbers. So there are $32 - 2 = 30$ ways of writing the numbers , so that each colour is used at least once.

(4). Solution: $3^5 - 3 = 243 - 3 = 240$.(Explain)

Problems (5) to (8): Do it yourself.

(9). Let us form a tabular column and look at the pattern generated.

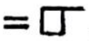
Stage No.	No of segments removed	No of segments remaining
S_0	0 $(2^0 - 1)$	1
S_1	1 $(2^1 - 1)$	2
S_2	$1+2=3$ $(2^2 - 1)$	$2^2 = 4$
S_3	$3+4=7$ $(2^3 - 1)$	$2^3 = 8$
S_4	15 $(2^4 - 1)$	$2^4 = 16$
S_5	31 $(2^5 - 1)$	$2^5 = 32$
S_6	63 $(2^6 - 1)$	$2^6 = 64$

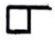
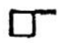
Stage	Length of the segments (unit)	
No.	remaining	removed
S_0	1	0
S_1	$\frac{2}{3}$	$1 - \frac{2}{3} = \frac{1}{3}$
S_2	$\frac{4}{9} = \frac{2^2}{3^2}$	$1 - \frac{4}{9} = \frac{5}{9}$
S_3	$\frac{8}{27} = \frac{2^3}{3^3}$	$1 - \frac{8}{27} = \frac{19}{27}$
S_4	$\frac{2^4}{3^4} = \frac{16}{81}$	$1 - \frac{16}{81} = \frac{65}{81}$
S_5	$\frac{2^5}{3^5}$	$1 - \frac{2^5}{3^5} = \frac{3^5 - 2^5}{3^5} = \frac{211}{243}$
S_6	$\frac{2^6}{3^6}$	$1 - \frac{2^6}{3^6} = \frac{3^6 - 2^6}{3^6} = \frac{665}{729}$

Generalise this for nth stage.

Problems (10) to (13): Do it yourself.

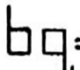
(14). a) (iii) 12 should be grouped as number of 5s and units; $12 = 2 \times 5 + 2$.

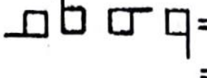
Since $2 =$ , $12 = 2 \times 5 + 2 =$ 

(*Note:* Rightmost  implies 2, the next  to the left implies 2×5.)

(v) $127 = 5 \times 5 \times 5 + 2 = 5^3 + 2 =$ 

(*Note:* No 5^2 or 5^1 terms here.)

b) (ii)  $= 1 \times 5 + 3 = 8$.

(v)  $= 4 \times 5^3 + 1 \times 5^2 + 2 \times 5 + 3$

$= 500 + 25 + 10 + 3 = 538$

c) Yes. You can write the given number as a sum of different powers of 5, so that you can find 'how many' different powers add up to the numbers.

If you write these numbers, using five symbols representing each of the powers of 5, (in decreasing powers of 5) you get the number. The right most symbol will be a number less than 5. [If it is zero, then the number is divisible by 5].

Problems (15) to (18): Do it yourself.

(19). Clearly $G = 1$. There are 6 other letters, A, B, E, L, M and S used and these must be assigned 6 different digits other than 1. Clearly $B \geq 5$.
Case(1): No carry over from the units place. Then $E + L = 5$ and $S + L = 10 + E$. (For, if $S + L = E$, then $E + L + L = E$ and hence $L = 0$. This means $E + L = E + S$. Not possible!)
$\therefore \quad E + L + L = 10 + E$ and hence $L = 5$.
Since $E + L$ gives no carry, $E < 5$. Let $E = 4$. The alphamatic reduces to

$$\begin{array}{r} B\ A\ 9\ 4 \\ B\ A\ 5\ 5 \\ \hline 1\ A\ M\ 4\ 9 \end{array}$$

We need to find A, B and M from 0, 2, 3, 6, 7, 8. Since $M \neq 1$ and 5, $A \neq 0$, 2 or 7.
If $A = 3$, then $M = 7$ and there is no carry from $A + A$. Hence $2B = 10 + A = 13$ odd numbers, not possible.
IF $A > 5$, then there is a carry from $A + A$ and hence $2B + 1 = 10 + A$ must be odd. But the only available digits for A are 6 and 8. Thus $E = 4$ does not give rise to a solution.

Let $E = 3$. The problem reduces to

$$\begin{array}{r} B\,A\,8\,3 \\ B\,A\,5\,5 \\ \hline 1\,A\,M\,3\,8 \end{array}$$

Now A, B, M can only take the digits 0, 2, 4, 6, 7, 9. Since $M \neq 1, 3, 5$, $A \neq 0, 2, 7$. If $A = 4$, then $M = 9$ is possible. This gives $2B = 10 + A = 14$ which gives $B = 7$.
$\therefore$ one solution is

$$\begin{array}{r} 7\;4\;8\;3 \\ 7\;4\;5\;5 \\ \hline 1\;4\;9\;3\;8 \\ \hline \end{array}$$

Try for the other choice $E = 2$.

Case (ii): There is a carry from the units place. *i.e.*, $E + L = 10 + S$. $S + L + 1 = E$ or $10 + E$. If $S + L + 1 = E$, then $S + L + 1 + L = 10 + S$ leading to $2L + 1 = 10$, not possible!
$\therefore$ $S+L+1 = 10+E$. This, when substituted in $E+L = 10+S$ gives, $S + L - 9 + L = 10 + S$ *i.e.*, $2L = 19$. Not possible! Hence Case (ii) gives rise to no solution. The only solution is

$$\begin{array}{r} 7\;4\;8\;3 \\ 7\;4\;5\;5 \\ \hline 1\;4\;9\;3\;8 \\ \hline \end{array}$$

Problems (20) to (29): Do it yourself.

(30). $a^2 + bc = ab + bd = ac + dc = bc + d^2 = 0$
$\Longrightarrow b(a+d)$ or $c(a+d)$
$= 0 \Longrightarrow$ (if $b, c, \neq 0$), $a = -d$

$$\left.\begin{array}{r} a^2 + bc = 0 \\ bc + d^2 = 0 \end{array}\right\} \Longrightarrow bc = -a^2 = -d^2$$

Thus fixing up a (or d), we can find the other numbers. $a = 10$ (say) gives $d = -10$, $bc = -100$, $b = 5$ $c = -20$ etc, or $a = -12$, gives $d = 12$, $bc = -144$, $b = -9$ $c = 16$ etc.

$$\text{Thus} \begin{pmatrix} 10 & 5 \\ -20 & -10 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -12 & -9 \\ 16 & 12 \end{pmatrix}$$

satisfy the requirements of the problem.

(31). Do it yourself.

(32).

$$\begin{pmatrix} 31 & 49 \\ 46 & 68 \end{pmatrix}, \begin{pmatrix} 38 & 56 \\ 24 & 39 \end{pmatrix} \qquad Z = 1, A = 26, dc = 2, -1, -1, 1$$

$$\begin{pmatrix} 31 & 49 \\ 46 & 68 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} (62-49) & -31+49 \\ (92-68) & -46+68 \end{pmatrix} = \begin{pmatrix} 13 & 18 \\ 24 & 22 \end{pmatrix}$$

A	B	C	D	E	F	G	H	I	J	K	L	M
26	25	24	23	22	21	20	19	18	17	16	15	14

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
13	12	11	10	9	8	7	6	5	4	3	2	1

So the message is NICE for the first matrix. For the second matrix we have

$$= \begin{pmatrix} 38 & 56 \\ 24 & 39 \end{pmatrix} \times \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 76-56 & -38+56 \\ 48-39 & -24+39 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & 18 \\ 9 & 15 \end{pmatrix} \Rightarrow \begin{pmatrix} G & I \\ R & L \end{pmatrix}$$

Taking the two matrices together, the message is NICE GIRL.

(33).

$$= 16\,875\,000\,000\,000 \times 390\,625$$
$$= 16\,875 \times 10^9 \times 390\,625$$
$$= 25 \times 675 \times 10^9 \times 25 \times 15\,625$$
$$= 5^2 \times 5^2 \times 27 \times 10^9 \times 5^2 \times 125 \times 125$$
$$= 5^2 \times 5^2 \times 3^3 \times 5^2 \times 10^9 \times 5^6$$
$$= 5^{21} \times 3^3 \times 2^9 \quad \text{(or)} \quad 2^9 \times 3^3 \times 5^{21}$$
$$\Rightarrow ICU \quad \text{(you can take it as "I see you".)}$$

Problems (34) to (41): Do it yourself.

(42). Separate two of the coins and keep them aside. Of the remaining 6 coins, place 3 coins on each pan.

(a) If they balance each other then, one of the two coins, kept aside is lighter, and the lighter can be found using the balance a second time.

(b) If one of the pans go up, (when 3 coins are placed on each pan), then one of the three coins placed on that pan is lighter. Now keep one coin aside, and find if one of the other two is lighter, using the balance. If they both are of equal weight, the coin kept aside is lighter.

(43). Do it your self